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FINITE STATISTICAL GAMES AND LINEAR PROGRAMMING

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ABSTRACT

The dual linear programs associated witb finite statistical games are investigated and their optimal solutions are interpreted. The usual statistical game is generalized to a two-sided (inference) game and its possible application as a tactical model is discussed.

1. INTRODUCTION

In this paper, we investigate the dual linear programs associated with finite statistical games (i.e., finite strategy and observation spaces) and interpret their optimal solutions. Furthermore, we generalize the usual (one-sided inference) statistical game to a two-sided (inference) statistical game where each player makes an initial strategy choice, partially implements his initial strategy while making an observation allowing him to infer about his opponent's initial choice, and finally makes a secondary choice restricted by the partial implementation of his initial strategy.

In Sections 2, 3, and 4, we present a hierarchy of three games and their associated dual linear programs. Each game in the hierarchy contains its predecessor as an imbedded special case. Section 2 treats the familiar Rectangular Game. Section 3 treats the Statistical Game of Statistical Decision Theory. Section 4 generalizes to the Two-Sided Statistical Game. Section 5 briefly discusses the Two-Sided Statistical Game as a tactical warfare model. All summations are over their full ranges unless otherwise specified.

2. RECTANGULAR GAME

Suppose we have two players, Blue and Red, with finite pure strategy sets indexed over I and J, respectively. Let a_{ij} be the payoff to Blue when he chooses pure strategy $i \epsilon I$ and Red chooses pure strategy $j \epsilon J$; we make the zero-sum assumption that $-a_{ij}$ is the corresponding payoff to Red. Let x_i denote the probability that Blue chooses pure strategy i, and let y_j be the probability that Red chooses pure strategy j. The Minimax Theorem assures the existence of a saddle point in mixed strategies. Moreover, we have the familiar dual linear programs

$$\max v \longrightarrow$$

(1) $$\text{s.t.} \sum_i a_{ij} x_i \geq v; \qquad j \epsilon J$$

$$\sum_i x_i = 1$$

$$x_i \geq 0; \qquad i \epsilon I$$

$$\min u \longrightarrow$$

$$\text{(2)} \qquad \text{s.t.} \sum_j a_{ij} y_j \leq u; \qquad i \in I$$

$$\sum_j y_j = 1$$

$$y_j \geq 0; \qquad j \in J$$

either of which can be solved to yield optimal strategies and the value of the game.

3. STATISTICAL GAME

Suppose that Red chooses first and that Blue makes one of a finite number of mutually exclusive and exhaustive observations indexed over K, thereby allowing him to infer about Red's choice before making his choice. Let p_{jk} denote the probability (assumed known) that Blue observes $k \in K$ given that Red has chosen pure strategy j. This game can obviously be formulated as a rectangular game where Blue has pure strategies of the form $\{i_k\}_{k \in K}$ implying his choice of $i_k \in I$ after observing k. The expected payoff to Blue is $\sum_k a_{i_k j} p_{jk}$ when he chooses pure strategy $\{i_k\}_{k \in K}$ and Red chooses pure strategy j. One can write down the analogs to (1) and (2) for this game, but trivial rearrangements and substitutions yield the equivalent dual linear programs

$$\max v \longrightarrow$$

$$\text{(3)} \qquad \text{s.t.} \sum_k \sum_i a_{ij} p_{jk} x_{ki} \geq v; \qquad j \in J$$

$$\sum_i x_{ki} = 1; \qquad k \in K$$

$$x_{ki} \geq 0; \qquad i \in I,\ k \in K$$

$$\min \sum_k u_k \longrightarrow$$

$$\text{(4)} \qquad \text{s.t.} \sum_j a_{ij} p_{jk} y_j \leq u_k; \qquad i \in I,\ k \in K$$

$$\sum_j y_j = 1$$

$$y_j \geq 0; \qquad j \in J$$

Suppose we denote the optimal solutions to (3) and (4) with stars. The x_{ki}^* represent Blue's optimal mixed strategy in behavioral form; i.e., x_{ki}^* is the probability that Blue chooses i after observing k. The optimal dual variables yield a nice interpretation for Blue. u_k^* represents a partial value of the game with respect to observation k, and $\sum_j y_j^* p_{jk}$ is the probability that k is observed. Hence, we can interpret $u_k^* / \sum_j y_j^* p_{jk}$ as the conditional (interim) value of the game to Blue given that he has observed k.

In conventional statistical decision theory, Blue represents the statistician and Red represents nature. A mixed strategy for Blue is called a "randomized decision rule" and his optimal strategy is called "maximin." A mixed strategy for Red is called an "a priori distribution" and his optimal strategy is called "least favorable." We remark that there exist situations [9] where the payoff depends on the observation (i.e., a_{ijk}). It should be clear that this generalization can be handled identically.

4. TWO-SIDED STATISTICAL GAME

We now assume that each player makes an initial choice, makes an observation allowing him to infer about his opponent's choice, and then makes a secondary choice from a restricted set of pure strategies. We use the previous notation for Blue, and we denote his admissible secondary strategy set by $M_i \subset I$ after initial choice i; we assume $i \epsilon M_i$. We assume that Red makes one of a finite number of mutually exclusive and exhaustive observations indexed over T, and we let q_{it} denote the probability that Red observes $t \epsilon T$ given that Blue has chosen i initially. We denote Red's admissible secondary strategy set by $N_j \subset J$ after initial choice j; we assume that $j \epsilon N_j$. This game can be formulated as a rectangular game with pure strategies of the form $(i, \{i_k\}_{k \epsilon K})$ with $i_k \epsilon M_i$ and $(j, \{j_t\}_{t \epsilon T})$ with $j_t \epsilon N_j$ for Blue and Red, respectively. The expected payoff to Blue is $\sum_k \sum_t a_{i_k j_t} p_{jk} q_{it}$ when he chooses $(i, \{i_k\}_{k \epsilon K})$ and Red chooses $(j, \{j_t\}_{t \epsilon T})$. The analogs to (1) and (2) reduce to the following dual linear programs.

$$\max v \longrightarrow$$

$$\text{(5)} \qquad \text{s.t.} \sum_i \sum_k \sum_{m \epsilon M_i} a_{mn} p_{jk} q_{it} x_{ikm} \geq v_{jt};\; n \epsilon N_j,\; t \epsilon T,\; j \epsilon J$$

$$\sum_t v_{jt} = v;\; j \epsilon J$$

$$\sum_{m \epsilon M_i} x_{ikm} = w_i;\; k \epsilon K,\; i \epsilon I$$

$$\sum_i w_i = 1$$

$$x_{ikm} \geq 0;\; m \epsilon M_i,\; k \epsilon K,\; i \epsilon I$$

$$\min u \longrightarrow$$

$$\text{(6)} \qquad \text{s.t.} \sum_j \sum_t \sum_{n \epsilon N_j} a_{mn} p_{jk} q_{it} y_{jtn} \leq u_{ik};\; m \epsilon M_i,\; k \epsilon K,\; i \epsilon I$$

$$\sum_k u_{ik} = u;\; i \epsilon I$$

$$\sum_{n \epsilon N_j} y_{jtn} = z_j;\; t \epsilon T,\; j \epsilon J$$

$$\sum_j z_j = 1$$

$$y_{jtn} \geq 0;\; n \epsilon N_j,\; t \epsilon T,\; j \epsilon J$$

Again, we have a behavioral representation of the optimal strategies. w_i^* is the probability that Blue chooses i initially, and if $w_i^* > 0$, x_{ikm}^*/w_i^* is his probability of subsequently choosing $m \epsilon M_i$ given that i was chosen initially and k was observed. Again, the optimal dual variables yield a nice interpretation for Blue. We can interpret $u_{ik}^*/\sum_j z_j^* p_{jk}$ as the conditional value of the game to Blue given that he chose i initially and observed k.

5. TACTICAL APPLICATION

The Two-Sided Statistical Game may be useful as a model for tactical warfare where both sides have reconnaissance capabilities. Suppose that a_{ij} represents Blue's subjective probability of mission accomplishment given that he selects i and Red selects j. The value of the game is then Blue's unconditional subjective probability of mission accomplishment. Zero-sum payoff represents the usual conservative "worst case" criterion.

A value can be assigned to any particular type of reconnaissance (e.g., aerial) by solving the game with and without it, the difference in values representing the increment of subjective probability of mission accomplishment contributed by the specified type of reconnaissance.

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