On the Superposition of Point Processes

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On the Superposition of Point Processes

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SUMMARY

Given point processes $N_1, \ldots, N_m$ their superposition is the point process N defined by $N(t) = N_1(t)+\ldots+N_m(t)$, $t \geqslant 0$. An equivalent description of the system $(N_1, \ldots, N_m)$ is by the process (X_n, T_n) where the T_n are the points of N, and $X_n = k$ if and only if T_n is a point of N_k. The use of (X, T) process enables one to study the dependence of $N_1, \ldots, N_m$. Necessary and sufficient conditions are obtained for $N_1, \ldots, N_m$ to be independent, and for the superposition N to be a renewal process. For example, if N_1 and N_2 are renewal processes and X is independent of N, then N is a renewal process only if X is a homogeneous Markov chain.

1. INTRODUCTION

CONSIDER a stochastic process $\{N(t);\ t \geqslant 0\}$ whose sample functions are right continuous, non-decreasing, integer-valued step functions with value zero at $t = 0$ and steps of magnitude one. Such a process can be considered as the counting process associated with a point process on $(0, \infty)$. In fact, we may refer to the N process as a point process on $(0, \infty)$. With this convention, we say that a point process is regular if $P\{N(t) < \infty\} = 1$ for all $t \in (0, \infty)$. Henceforth, all processes mentioned are assumed to be regular. The consecutive jump times of the N process are denoted by $T_1, T_2, \ldots$ and these will be referred to as the "points" of N. The random variables $T_1, T_2-T_1, T_3-T_2, \ldots$ will be called the "intervals" of N. If the intervals of N are independent and are identically distributed (except possibly for the first interval), then N is called a renewal process. A renewal process is called persistent if

$$P\{T_1 < \infty\} = P\{T_2 - T_1 < \infty\} = 1.$$

All renewal processes mentioned hereafter are taken to be persistent. A point process N is called periodic (or arithmetic) if the points of N occur only at multiples of some $\omega > 0$ with probability one. A renewal process with period one is called a recurrent process.

We shall call any finite collection of point processes a system of point processes (denoted by $(N_1, \ldots, N_\nu)$). A system is called simple if any coincidence of points from two or more component processes occurs with probability zero. All systems in this paper are assumed to be simple. A system is independent if the component processes are independent. Otherwise, the system is dependent. The process N defined by

$$N(t) = \sum_{k=1}^{\nu} N_k(t) \tag{1}$$

for $t \geqslant 0$ is a point process called the superposition of the system $(N_1, \ldots, N_\nu)$.

Let $(N_1, \ldots, N_\nu)$ be a system of point processes with superposition N. Let $T_1, T_2, \ldots$ be the points of N and define $X_n = k$ if and only if T_n is a point in the N_k process. The (X, T) process is probabilistically equivalent to the system $(N_1, \ldots, N_\nu)$ and we refer to the X process as the indicator process of the system. One might think of the indicator process as the means of decomposing the superposed process to obtain the system.

In the next section, we investigate the superposition of a system of point processes. We shall be particularly interested in the case where the superposition and the indicator process are independent. In this case, the indicator process corresponds to our intuitive notion of a decomposition rule where points are assigned without regard to spacing.

Virtually all work concerning the superposition problem has been confined to independent systems of point processes (see, for example, Palm (1943), Cox and Smith (1954), Khinchine (1960)). The independence assumption would seem to limit both the generality and applicability of the results obtained. Superposition problems involving dependent systems arise frequently in applications such as traffic and queueing networks so that more extensive investigations are warranted by need. The indicator process provides a means of characterizing dependence.

2. Superposition Theorems

In this section we give several theorems which illustrate the role played by the indicator process. We shall deal only with systems of two processes (2-systems) although some of the results can be generalized to encompass larger systems.

By a Bernoulli process on $\{a,b\}$, we mean a sequence $\{X_n;\, n\geqslant 1\}$ of independent and identically distributed random variables with

$$P\{X_1 = b\} = 1-P\{X_1 = a\} = p = 1-q\in(0,1).$$

Initially, we prove the following characterization for the Poisson distribution.

Lemma 1. Let $X_0\equiv 0$ and let $\{X_n : n\geqslant 1\}$ be a Bernoulli process on $\{0,1\}$. Let N be a non-negative integer valued random variable which is independent of the X process. Let $U = X_0+X_1+\ldots+X_N$ and let $V = N-U$. Then, U and V are independent if and only if N is Poisson distributed.

Proof. The sufficiency part of the lemma is easy to prove and is well known. To prove the necessity, we suppose that U and V are independent. Let $g(z) = E(z^N)$ for $|z|\leqslant 1$. Clearly, $E(z^U) = g(q+pz)$ and $E(z^V) = g(p+qz)$ for $|z|\leqslant 1$. By independence, we have that

$$g(z) = g(q+pz)\,g(p+qz) \tag{2}$$

and it is easy to see that this implies that g is not a polynomial of finite degree unless $P\{N = 0\} = 1$, in which case we take N to be Poisson distributed with parameter zero. For $P\{N = 0\}<1$, it follows that there exists no finite integer n_0 such that $P\{N\leqslant n_0\} = 1$. Consequently, $P\{U = k\}>0$ and $P\{V = k\}>0$ for all $k = 0,1,2,\ldots$ and the proof is now completed by applying Theorem 1 of Chatterji (1963).

Remark. One can prove lemma 1 without reference to Chatterji's theorem by showing that $h = (d/dz)\ln g$ is necessarily constant on $(0,1)$ for (2) to be satisfied.

Theorem 2. Let (N_1,N_2) be a system of point processes with indicator process X and superposition N. Suppose that X is a Bernoulli process, independent of N. Then, the system is independent only if

$$P\{N(t) = k\} = \exp\{-\lambda(t)\}\frac{[\lambda(t)]^k}{k!}\,(t\geqslant 0);\quad (k = 0,1,\ldots) \tag{3}$$

for some non-decreasing function λ on $[0,\infty)$ with $\lambda(0) = 0$.

Proof. It follows from lemma 1 that $N(t)$ is Poisson distributed for every $t \geqslant 0$. Also, $\lambda(0) = 0$ by definition and it is obvious that λ is non-decreasing.

Lemma 3. Let (N_1, N_2) be a system of Poisson processes with indicator process X and superposition N. If X is a Bernoulli process, independent of N, then N is Poisson if and only if the system is independent.

Proof is obvious.

Theorem 4. Let (N_1, N_2) be a system of Poisson processes with indicator process X and superposition N. Suppose that X is a stationary Markov chain, independent of N, and that the rates of N_1 and N_2 are not equal. Then, N is a renewal process if and only if the system is independent.

Proof. The sufficiency is obvious. To prove the necessity, suppose that N is a renewal process and let $G(x) = P\{T_2 - T_1 \leqslant x\}$ where $T_1, T_2, \ldots$ are the points of N. Let $\lambda_1 > 0$ and $\lambda_2 > 0$ be the rates of N_1 and N_2 respectively. By assumption $\lambda_1 \neq \lambda_2$. Let $P\{X_{n+1} = 1 \mid X_n = 1\} = p$ and $P\{X_{n+1} = 2 \mid X_n = 2\} = q$ $(n \geqslant 1)$ where p, $q \in [0, 1]$. Let F_j be the generating function of the recurrence time distribution of state j. For $|z| \leqslant 1$, we have

$$\{1 - F_1(z)\}(1 - qz) = \{1 - F_2(z)\}(1 - pz). \tag{4}$$

Let

$$g(s) = \int_0^\infty \exp(-sx)\, G\{dx\} \quad \text{for} \quad R\exp(s) \geqslant 0.$$

Then, we must have

$$F_j\{g(s)\} = \frac{\lambda_j}{\lambda_j + s} \quad (j = 1, 2). \tag{5}$$

From (4) and (5), we get

$$\{p\lambda_1 - q\lambda_2 + (p - q)s\}\, g(s) = \lambda_1 - \lambda_2. \tag{6}$$

Since $g(0) = 1$, it follows that

$$(1 - q) = \frac{\lambda_1}{\lambda_2}(1 - p). \tag{7}$$

Upon differentiating in (6) and setting $\mu = -g'(0)$, we have

$$\mu(\lambda_1 - \lambda_2) = p - q \tag{8}$$

and further, from the elementary renewal theorem,

$$\frac{1}{\mu} = \lambda_1 + \lambda_2. \tag{9}$$

Solving for p and q from (7), (8) and (9), we get

$$p = (1 - q) = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

so that X is a Bernoulli process. From (6)

$$g(s) = \frac{\lambda}{\lambda + s}$$

where $\lambda = \lambda_1 + \lambda_2$, which implies that N is Poisson and the assertion now follows from lemma 3.

Remark. If $\lambda_1 = \lambda_2$ and $p = \frac{1}{2}$ or $q = \frac{1}{2}$, the theorem holds: from (7), then, $p = q = \frac{1}{2}$ and then $p = (1-q) = \frac{1}{2}$.

We now give the following characterization for the geometric distribution.

Lemma 5. Let $Y_1, Y_2, \ldots$ be a sequence of independent and identically distributed negative exponential random variables. Let N be a positive integer valued random variable which is independent of the Y sequence. Then $U = Y_1 + Y_2 + \ldots + Y_N$ is negative exponentially distributed if and only if N has the geometric distribution.

Proof. Let $g(z) = E(z^N)$ for $|z| \leqslant 1$ and let $\mu = 1/E(Y_1) > 0$. Clearly, for

$$R\exp(s) \geqslant 0, E\{\exp(-sU)\} = g\{\mu/(\mu+s)\}.$$

If $P\{N = n\} = pq^{n-1}$ $(n \geqslant 1)$ for $p = 1-q \in (0,1]$, then $g(z) = pz/(1-qz)$ so

$$E\{\exp(-sU)\} = p\mu/(p\mu+s).$$

Conversely, if U is negative exponentially distributed, with parameter $\lambda \in (0, \mu]$, then $g\{\mu/(\mu+s)\} = \lambda/(\lambda+s)$ for $R\exp(s) \geqslant 0$ and it follows that $g(z) = pz/(1-qz)$ with $p = \lambda/\mu$.

Theorem 6. Let (N_1, N_2) be a system of point processes with indicator process X and superposition N. If N_1 is Poisson and X is independent of N, then N is Poisson if and only if N_2 is Poisson and the system is independent.

Proof. The sufficiency is obvious. To prove the necessity, let N_1 be Poisson with rate $\lambda > 0$ and let N be Poisson with rate $\mu \geqslant \lambda$. Then, M intervals from the N process will form one interval in the N_1 process and it follows from lemma 5 that M is geometric with $p = \lambda/\mu$. If $p = 1$, we can interpret N_2 as a Poisson process with rate zero and the assertion follows trivially. If $p < 1$, X is a Bernoulli process and the assertion follows from lemma 3 as N_2 is clearly Poisson with rate $\mu - \lambda$.

We will now prove a characterization theorem for the indicator process associated with a 2-system of point processes. We take $\mathscr{T}_1 = \{T_1', T_2', \ldots\}$ and $\mathscr{T}_2 = \{T_1'', T_2'', \ldots\}$ to be the points of a system (N_1, N_2) where N_1 and N_2 are periodic with period one and

$$P\{\mathscr{T}_1 \cap \mathscr{T}_2 = \phi\} = P\{\mathscr{T}_1 \cup \mathscr{T}_2 = J\} = 1,$$

ϕ being the empty set and J the positive integers. We let X be the indicator process, and $\{Y_1, Y_2, \ldots\}$ is taken to be a sequence of independent and identically distributed (except perhaps for the distribution of Y_1) positive random variables independent of the X process. Define $S_n = Y_1 + Y_2 + \ldots + Y_n$. In this context, the following lemma provides a characterization for two state Markov chains.

Lemma 7. N_1 and N_2 are recurrent processes if and only if X is a homogeneous Markov chain.

Proof. The sufficiency is well known. To prove the necessity, consider the event $\{X_n = j_n, \ldots, X_1 = j_1\}$, and let $\{n_1, \ldots, n_k\} = \{m : j_m = j_n\}$, where $n_1 \leqslant \ldots \leqslant n_k = n$. Then $\{X_n = j_n, \ldots, X_1 = j_1\} = \{T_k = n, \ldots, T_1 = n_1\}$ where $T_k = T_k'$ or T_k'', according to whether $j_n = 1$ or 2. Thus

$$P\{X_{n+1} = j \mid X_n = j_n, \ldots, X_1 = j_1\} = \begin{Bmatrix} P\{T_{k+1} = n+1 \mid T_k = n, \ldots, T_1 = n_1\}, & j = j_n \\ 1 - P\{T_{k+1} = n+1 \mid T_k = n, \ldots, T_1 = n_1\}, & j \neq j_n \end{Bmatrix},$$

and since N_1 and N_2 are taken to be recurrent,

$$P\{T_{k+1} = n+1 \mid T_k = n, \ldots, T_1 = n_1\} = P\{T_{k+1} = n+1 \mid T_k = n\} = P\{X_{n+1} = j_n \mid X_n = j_n\}$$

so that X is a Markov chain and it is clearly homogeneous, so the proof is complete.

Lemma 8. Let $\{T_1, T_2, \ldots\}$ be either $\mathscr{T}_1$ or $\mathscr{T}_2$. Then, $S_{T(1)}, S_{T(2)}, \ldots$ are points of a renewal process if and only if $T_1, T_2, \ldots$ are the points of a recurrent process.

Proof. The sufficiency is trivial. To prove the necessity, suppose that $S_{T(1)}$, $S_{T(2)} - S_{T(1)}, \ldots$ are independent. Let I be a non-empty, finite subset of the set of positive integers. Writing $U_i = S_{T(i+1)} - S_{T(i)}$, with $R\exp(\sigma_i) \geqslant 0$, $i \in I$, we have

$$E\left\{\exp\left(-\sum_{i\in I} \sigma_i U_i\right)\right\} = E\left[E\left\{\exp\left(-\sum_i \sigma_i U_i\right) \middle| T_{i+1} - T_i, \quad i \in I\right\}\right]$$

$$= E\left\{\prod_{i\in I} z_i^{T_{i+1}-T_i}\right\}$$

where $z_i = E\{\exp(-\sigma_i Y_2)\}$, $i \in I$. By the assumption of independence,

$$E\left\{\exp\left(-\sum_{i\in I} \sigma_i U_i\right)\right\} = \prod_{i\in I} E\{\exp(-\sigma_i U_i)\} = \prod_{i\in I} E\{z_i^{T_{i+1}-T_i}\}.$$

For any $i \in I$, z_i takes on at least every value in $(0, 1)$ and $|z_i| \leqslant 1$. Hence,

$$E\left\{\prod_{i\in I} z_i^{T_{i+1}-T_i}\right\} = \prod_{i\in I} E\{z_i^{T_{i+1}-T_i}\}$$

for all $z_i \in (0, 1)$, $i \in I$; and it follows that $T_2 - T_1, T_3 - T_2, \ldots$ are mutually independent. Obviously, the same proof with a few trivial variations can be used to show that $T_1, T_2 - T_1, T_3 - T_2, \ldots$ are mutually independent. Furthermore, if U_i and U_j are identically distributed, $(i, j \in I)$, then we have

$$E\{z_i^{T_{i+1}-T_i}\} = E\{z^{T_{j+1}-T_j}\}$$

for all $z \in (0, 1)$ so that $T_{i+1} - T_i$ and $T_{j+1} - T_j$ are identically distributed, and this completes the proof.

We now summarize the results of lemmas 7 and 8 in the following theorem.

Theorem 9. Let (N_1, N_2) be a system of point processes with indicator process X and superposition N. If N_1 and N_2 are renewal processes and X is independent of N, then N is a renewal process only if X is a homogeneous Markov chain.

Proof. Let $Y_1, Y_2, \ldots$ be the intervals of N. Let

$$\mathscr{T}_1 = \{n : X_n = 1\} \quad \text{and} \quad \mathscr{T}_2 = \{n : X_n = 2\}.$$

$S_{T'(1)}, S_{T'(2)}, \ldots$ and $S_{T''(1)}, S_{T''(2)}, \ldots$ are the points of N_1 and N_2 respectively and are thus points of renewal processes. Hence by lemma 8, $T_1', T_2', \ldots$ and $T_1'', T_2'', \ldots$ are points of recurrent processes, and it follows from lemma 7 that X is a homogeneous Markov chain.

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