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AN APPLICATION OF CHANCE CONSTRAINED PROGRAMMING TO PORTFOLIO SELECTION IN A CASUALTY INSURANCE FIRM*†

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The problem of portfolio selection is discussed with special emphasis on the casualty insurance firm. A single period optimization model is developed in which expected return is maximized subject to a chance constraint requiring return to be greater than some lower bound with a stipulated probability. It is demonstrated that this approach provides an operational means of selecting a Baumol efficient portfolio. Additional chance constraints are used to maintain the firm's liquidity. The evaluation of optimal portfolios is discussed and the evaluators for the portfolio model are developed. Finally, an example is provided.

The problem considered in this paper is the selection of a Baumol efficient portfolio [2] for a casualty insurance firm through the use of the technique of chance constrained programming developed by Charnes and Cooper (see [4], [5], [6], [7]). It is hoped that some of the problems dealt with here will suggest ways of handling similar problems in other settings.[1]

Section 1 provides an introduction to the subject of optimal portfolio selection and a review of the salient characteristics of the investment operations in the casualty insurance industry. In Section 2 we introduce our portfolio model. Section 3 discusses the important topic of evaluation of optimal portfolios and Section 4 provides an example. Finally, two appendices provide: (A) a discussion of chance constraints; and (B) a summary of the evaluators.

1. Introduction

Assuming that the rates of return associated with the firm's portfolio assets are random variables whose joint distribution is multivariate normal, Markowitz [14] developed the concept of an efficient portfolio in terms of the expected return and standard deviation of return. In Figure 1.1 such portfolios, said to be (E, σ) efficient, form the lower boundary of the set of attainable portfolios, $\mathcal{C}$, between A and B. However, in the absence of any information about the investor's preference structure it cannot be determined which of this subset of portfolios is preferred.

Baumol [2] proposed the replacement of the Markowitz (E, σ) criteria with an $(E, E - K\sigma)$ criteria, where $E - K\sigma$ is a measure of the risk involved in a given portfolio, and K depends upon the investor's attitude toward risk. As Baumol demon-

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[1] See Naslund and Whinston [15] and Charnes and Thore [8] for other applications of chance constrained programming to financial budgeting.

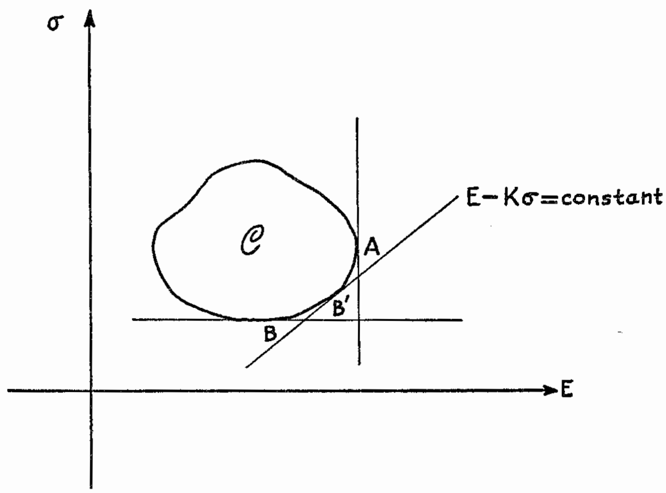

FIGURE 1.1

strates, this amounts to a reduction of the Markowitz efficient set from the boundary of $\mathcal{C}$ between A and B to the boundary of $\mathcal{C}$ between A and B' (see Figure 1.1).

In the present paper the firm maximizes expected return subject to a chance constraint requiring return to be greater than some lower bound with a stipulated probability. This provides an operational means of selecting a portfolio which is Baumol efficient with respect to that stipulated probability.

To see this, consider the chance constraint on loss,

$$P(R'x < t) \leqq \alpha,$$

where $R = (R_1, \cdots, R_n)'$ is a column vector of rates of return on the n possible assests, $x = (x_1, \cdots, x_n)'$ is the amount of each asset valued at purchase price (the decision variable), t is the specified value of the lower bound on the portfolio's return and α is the stipulated probability. The rates of return are assumed to be normally distributed with mean vector μ and positive definite covariance matrix Γ.

This chance constraint can be converted to a certainty equivalent constraint (see Appendix A) of the form

$$\mu'x - K(\alpha)(x'\Gamma x)^{1/2} \geqq t,$$

where K is the negative inverse function of the standard normal distribution function. Maximization of expected return subject to a chance constraint on loss is thus equivalent to maximization of expected return subject to $E - K\sigma$ greater than or equal to some lower bound. Thus, the model provides a means of selecting one of Baumol's efficient portfolios. Figure 1.2 provides an illustration. The set of feasible portfolios, $\mathcal{C}$, is recued to the shaded area by the chance constraint. The optimal portfolio will be D representing maximum expected return for $E - K\sigma \geqq t$. This is a Baumol efficient portfolio as it is on the boundary of $\mathcal{C}$ between A and B'.

The feasible set of portfolios is further restricted by chance constraints arising from the highly stochastic nature of operations in a casualty insurance company. These operations involve a random demand for cash which may arise from an excess of claims over premium income (see [10], p. 426). In order to maintain the firm's liquid-

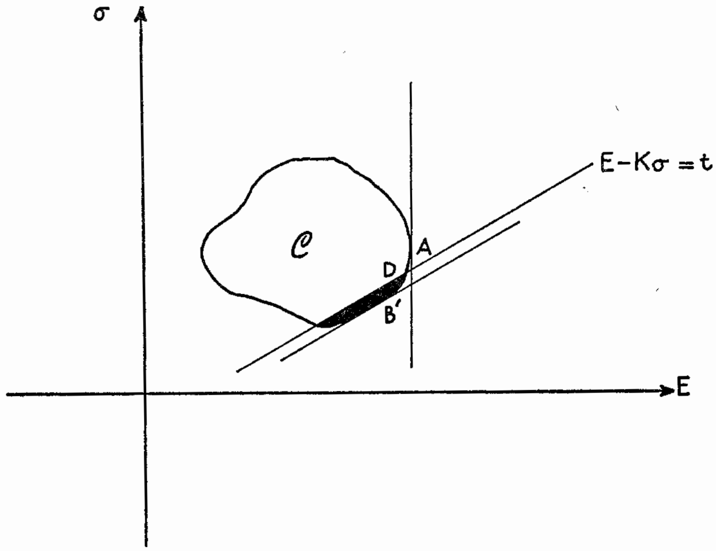

FIGURE 1.2

ity, we impose constraints on cash reserves and also on two ratios, the policyholder's surplus—premium ratio and the common stock—policyholder's surplus ratio.

By premiums we might mean either earned premiums or written premiums. "Insurance exposure" is defined as the ratio of *earned* premiums to policyholder's surplus. "The concept is significant because it serves as a rough measure of the financial capacity of the insurer to absorb unexpected underwriting losses" ([10], p. 415). Note that we are placing a lower bound on the reciprocal of the insurance exposure rather than an upper bound on insurance exposure (see [11], p. 52). Alternatively the Kenney rule limits the volume of premiums that may safely be *written* to not more than twice the total of policyholder's surplus (see [13], p. 443). The particular ratio of two to one (based primarily on Kenney's study of casualty failures in the great depression) is, of course, arbitrary. The point to be made is that the ratio, or its reciprocal, is taken by the industry as a measure of a financial soundness.

The common stock-policyholder's surplus ratio is also used as a measure of liquidity or financial soundness. The industry either places an upper bound on this ratio ([11], p. 52) or a lower bound on the ratio of liquid assets (including cash plus such quasi-cash items as high grade bounds with essentially no price fluctuations) to liabilities.[2]

There are other measures of liquidity or financial well being that could be incorporated into the constraints of our model instead of or in addition to those outlined above (see [11], p. 52). However, we are interested here in an approach to the investment problem. Thus, we consider what appear to be the most important characteristics and leave it to the reader to make what ever modifications he thinks important

2. Development of the Model

We suppose that there exists a collection of n noncash assets from which the firm is to select its portfolio for the ensuing investment period. The first $m \leqq n$ of these

[2] Since liquid assets are roughly equal to assets minus common stock and liabilities are equal to assets minus policyholder's surplus these are equivalent formulations. See Kulp [13], p. 445 and Davis [9], p. 1112.

assets are assumed to be common stocks and all assets are assumed to be available in unlimited amounts. It is supposed that the firm's market transactions take place instantaneously at the beginning of the period.

We denote by $b_i \geqq 0$ the pre-transaction amount of asset i held by the firm. The units are dollars based on market values at the beginning of the period. We let $x_i \geqq 0$ represent the post-transaction amount of asset i held by the firm in the same units. We shall refer to the vector $x = (x_1, \cdots, x_n)'$ as a "portfolio." (Prime indicates transposition.) $d_i \geqq 0$ is the dividend or equivalent fixed percentage due from asset i during the period based on market values at the beginning of the period. R_i denotes the percentage increase in value for asset i over the period based on market values at the beginning of the period. We suppose that $R = (R_1, \cdots, R_n)'$ is a vector random variable whose known probability distribution function is multivariate normal with $E(R) = \mu = (\mu_1, \cdots, \mu_n)'$ and cov $(R) = \Gamma = [\text{cov}(R_i, R_j)]$.

We let $b_0 \geqq 0$ represent the pre-transaction level of cash reserves and $b_0^* \geqq 0$ denotes a lower bound on cash reserves. The firm's net demand for cash during the period is denoted by D. It is supposed that D is a random variable whose known probability distribution function is univariate normal with $E(D) = \nu$ and var $(D) = \tau^2$. We assume that R and D are uncorrelated.

The criterion of optimization is taken to be the maximization of expected gain. That is, our objective is to determine an optimal portfolio x^* which maximizes

$$(1) \qquad E\left(\sum_{i=1}^{n} (R_i + d_i)x_i\right) = \sum_{i=1}^{n} (\mu_i + d_i)x_i$$

over all feasible portfolios.

The set of feasible portfolios is delineated by a number of constraints reflecting some of the special attributes of a casualty insurance firm. No specified return on an investment in bonds and securities is certain. The investor must consider the probability that his return will be below some acceptable minimum level. Thus, the model incorporates a generalized risk chance constraint (2) specifying that the end of period gain will fall below γ with a probability not to exceed α_1.

$$(2) \qquad \Pr\left\{\sum_{i=1}^{n} (R_i + d_i)x_i < \gamma\right\} < \alpha_1 .$$

The relationship of this constraint to Baumol's formulation has been discussed in the Introduction.

Two indicators of liquidity influence the behavior of the casualty insurance firm—the surplus-premium ratio and the common stock-surplus ratio ([9], [10], [11] and [15]). These have been incorporated into one period risk constraints.

$$(3) \qquad \Pr\left\{s + \sum_{i=1}^{m} R_i x_i + \sum_{i=1}^{n} d_i x_i - D < \lambda_p\right\} \leqq \alpha_2 .$$

This chance constraint specifies that the end of period surplus-premium ratio will fall below λ with a probability not to exceed α_2, where s is the policyholder's surplus recorded in the firm's accounts at the beginning of the period, and p is the annual rate of premium income, which is assumed to be constant throughout the period. Fluctuations in the market values of assets other than common stocks do not affect the surplus account as it is assumed that all other noncash assets are carried in the firm's accounts at some standardized book value.

The other self-imposed liquidity constraint specifies that the common stock-surplus ratio must not exceed δ:

$$(4) \qquad \sum_{i=1}^{m} x_i \leqq \delta s.$$

δ is a function of institutional factors which place an upper bound on the proportion of the portfolio held in the form of common stock; that this proportion is related to the surplus position of the firm is an historical practice of the industry ([13], p. 445 and [11], p. 52).

Most firms face a random demand for cash. The nature of this demand will affect the composition of their portfolio, both in terms of type and the proportions of noncash assets they hold. In the case considered here, it is through this demand for cash that the insurance operations of the firm influence the investment operations. In recent years the uncertainty and variability of the court settlements in accident suits has reduced the predictability of the demand of policy holders on the cash reserves of the casualty insurance firm. Because of this unpredictability it is important to include cash demand as a random variable in the optimization model.

$$\Pr\{b_0 - \textstyle\sum_{i=1}^{n}(x_i - b_i) - D < b_0^*\} \leqq \alpha_3 . \tag{5}$$

This chance constraint requires that the end of period level of cash reserves will fall below b_0^* with a probability not to exceed α_3.

We further require that the firm's net investment into noncash assets at the beginning of the period will not exceed the amount of excess cash, $b_0 - b_0^*$.

$$\textstyle\sum_{i=1}^{n}(x_i - b_i) \leqq b_0 - b_0^*. \tag{6}$$

This is merely a budget constraint.

Finally we include a simple nonnegativity requirement.

$$x_i \geqq 0, \qquad 1 \leqq i \leqq n. \tag{7}$$

We wish to make two observations. First, an additional chance constraint dealing with the end of period stock-surplus ratio could be introduced. However, it is felt that such a constraint would be an unnecessary restriction on appreciation in common stock values. Secondly, there may exist some additional legal constraints of the form $x_i \geqq b_i^* > 0$ or $\sum_{i \in I} x_i \geqq \rho > 0$, but such additions to the model would be trivial and we shall consequently ignore them.

At this point we wish to convert the chance constraints to their equivalent forms (see [4], [5], [6], [7]) and to restate the problem in vector notation (see Appendix A). To this end, the following notation is introduced. Let $c_i = \mu_i + d_i$ for $1 \leqq i \leqq n$ and let $c = (c_1, \cdots, c_n)'$. Let $c^* = (c_1, \cdots, c_m, d_{m+1}, \cdots, d_n)'$. Let $\epsilon = (1, \cdots, 1)'$ and $\epsilon^* = (1, \cdots, 1, 0, \cdots, 0)'$, the last $n - m$ elements of ϵ^* being zeros. Similarly, $b = (b_1, \cdots, b_n)'$. If

$$\Gamma = \left[\begin{array}{c|c} \Gamma_1 & \Gamma_2 \\ \hline \Gamma_3 & \Gamma_4 \end{array}\right],$$

where Γ_1 is $m \times m$, we let

$$\Gamma^* = \left[\begin{array}{c|c} \Gamma_1 & 0 \\ \hline 0 & 0 \end{array}\right].$$

Φ is taken to be the standard normal probability distribution function, i.e.,

$$\Phi(z) = \int_{-\infty}^{z} (1/\sqrt{2\pi}) \exp[-(\xi^2/2)]\, d\xi$$

and we let $K = \Phi^{-1}$. That is, K is the inverse function for the one-to-one function Φ.

Converting the chance constraints to their certainty equivalents we can state the problem formally in the following fashion:

$$\max_x f(x) = c'x$$

subject to:

$$\begin{aligned} g_1(x) &= -\gamma + c'x + K(\alpha_1)\sqrt{x'\Gamma x} \geqq 0 \\ g_2(x) &= s - \lambda p - \nu + c^{*\prime}x + K(\alpha_2)\sqrt{x'\Gamma^* x + \tau^2} \geqq 0 \\ g_3(x) &= b_0 - b_0^* - \epsilon'(x - b) + \min(0, -\nu + K(\alpha_3)\tau) \geqq 0 \\ g_4(x) &= \delta s - \epsilon^{*\prime}x \geqq 0 \\ x &\geqq 0. \end{aligned} \tag{8}$$

Note that the original constraints (5) and (6) have been consolidated into $g_3(x) \geqq 0$. The problem can be simplified analytically by the introduction of auxiliary spacer variables (see [7]). The resulting problem is a nonlinear programming problem in which a linear functional is minimized over a convex set, a problem for which a variety of computation algorithms are available.

3. Evaluation

In this section we assume that the optimal solution x^* has been determined. At the same time we obtain a vector of Kuhn-Tucker multipliers or shadow prices, $\omega^* = (\omega_1^*, \cdots, \omega_4^*)' \geqq 0$ corresponding to $g_i(x^*) \geqq 0$ for $i = 1, \cdots, 4$. Under general conditions the shadow price ω_i^* represents the rate at which the criterion function increases per unit relaxation of the constraint, $g_i(x^*) \geqq 0$. Clearly, if $g_i(x^*) > 0$, it must be that $\omega_i^* = 0$ since the constraint is not active at the point x^*. However, if $\omega_i^* > 0$, there is a possibility of increased expected gain through the alteration of stipulated parameters corresponding to the ith constraint (see [1] [15]).

Let u_γ^* be the rate of increase in the criterion function per unit increase in the parameter γ. We refer to u_γ^* as the "evaluator" for γ corresponding to the optimal solution x^*. The evaluator will be the product of the shadow price ω_i^* and the rate at which a unit increase in the parameter γ effects the constraint $g_i(x^*)$ summed over the constraints in which the parameter γ appears; that is,

$$u_\gamma^* = \textstyle\sum_{i=1}^4 \omega_i^* \, \partial g_i(x^*, \gamma)/\partial\gamma. \tag{9}$$

Evaluators for the other stipulated parameters are defined analogously.

In Appendix B, we have summarized the evaluators for the stimulated parameters in the model. The determination of these evaluators is perhaps more important than the calculation of the optimal portfolio x^*. It is possible that many firms operate in a nearly optimal manner on the basis of experience and common sense. However, the firm usually has no way of knowing the opportunity costs associated with selfimposed restrictions.

4. Example

We assume that the firm can choose between asset 1, a common stock, and asset 2, a bond. The following data are given (asset units are millions of dollars, i.e., $\$ \times 10^6$): $b_0 = 100$ and $b_0^* = 80$. $b' = (b_1, b_2) = (60{,}240)$. $\nu = 0$ and $\tau^2 = 100$. $\mu' = (\mu_1, \mu_2) = (.08, .04)$, $d' = (d_1, d_2) = (.02, 0)$, and

$$\Gamma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} 10^{-2} & 0 \\ 0 & 10^{-4} \end{bmatrix}.$$

Hence, $c' = (c_1, c_2) = (.10, .04)$, $c^{*\prime} = (c_1, 0) = (.10, 0)$, and

$$\Gamma^* = \begin{bmatrix} \sigma_{11} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 10^{-2} & 0 \\ 0 & 0 \end{bmatrix}.$$

$s = 100$ and $p = 300$. We set $\alpha_1 = \alpha_2 = \alpha_3 = .0228$ so that $K(\alpha_1) = K(\alpha_2) = K(\alpha_3) = -2$. The resulting problem can now be stated in terms of the remaining stipulated parameters.

$$\max_x f(x) = .10x_1 + .04x_2$$

subject to:

$$g_1(x) = -\gamma + .10x_1 + .04x_2 - 2(10^{-2}x_1^2 + 10^{-4}x_2^2)^{1/2} \geqq 0$$

$$g_2(x) = 100 - 300\lambda + .10x_1 - 2(10^{-2}x_1^2 + 100)^{1/2} \geqq 0$$

$$g_3(x) = 300 - x_1 - x_2 \geqq 0$$

$$g_4(x) = 100\delta - x_1 \geqq 0$$

$$x \geqq 0.$$

We now investigate three different configurations of the remaining stipulated parameters: γ, λ, and δ.

(1) Case I: $\gamma = 0$, $\lambda = .20$, $\delta = .50$.

The optimal portfolio is $x^* = (x_1^*, x_2^*)' = (50, 250)'$. The expected gain from the optimal portfolio is 15. We have that $g_1(x^*) > 0$, $g_2(x^*) > 0$, $g_3(x^*) = 0$, and $g_4(x^*) = 0$. From the Kuhn-Tucker conditions we have that $\omega_1^* = \omega_2^* = 0$ and that

(i) $.10 - \omega_3^* - \omega_4^* = 0$

(ii) $.04 - \omega_3^* = 0.$

Hence, $\omega_3^* = .04$ and $\omega_4^* = .06$. Using these results we compute $u_\gamma^* = 0$, $u_{\alpha_1}^* = 0$, $u_\lambda^* = 0$, $u_{\alpha_2}^* = 0$, $u_{b_0^*}^* = -.04$, $u_{\alpha_3}^* = 7.4$, and $u_\delta^* = 6$. We conclude that expected gain increases at a rate of .04 per unit decrease in b_0^*, .074 per hundredth of a unit increase in α_3, and .06 per hundredth of a unit increase in δ.

(2) Case II: $\gamma = 0$, $\lambda = .275$, $\delta = .80$.

The optimal portfolio is $x^* = (75,225)'$, and the resultant expected gain is 16.5. We have that $g_1(x^*) > 0$, $g_2(x^*) = 0$, $g_3(x^*) = 0$, and $g_4(x^*) > 0$. Hence, $\omega_1^* = \omega_4^* = 0$ and

(i) $.10 - .02\,\omega_2^* - \omega_3^* = 0$

(ii) $.04 - \omega_3^* = 0.$

Thus, $\omega_2^* = 3$ and $\omega_3^* = .04$ so that $u_\gamma^* = 0$, $u_{\alpha_1}^* = 0$, $u_\lambda^* = -900$, $u_{\alpha_2}^* = 695$, $u_{b_0^*}^* = -.04$, $u_{\alpha_3}^* = 7.4$, and $u_\delta^* = 0$. We have the startling result that the expected gain increases at a rate of .9 per thousandth of a unit decrease in λ and .695 per thousandth of a unit increase in α_2. We must bear in mind that the evaluators are instantaneous rates, e.g., u_λ^* is exact only over a decrease of $d\lambda$ and is an approximation for anything larger. Furthermore, even as approximations these rates apply only over intervals where the initially inactive constraints remain inactive; e.g., λ can only be decreased by one-third of a thousandth in this example before the fourth constraint becomes active. Nevertheless, the evaluators are easily obtained, and when properly interpreted, they provide a way of determining relevant opportunity costs.

(3) Case III: $\gamma = 4$, $\lambda = .20$, $\delta = .80$.

The optimal portfolio is $x^* = (48.5, 251.5)'$, and the resultant expected gain is 14.91. We have that $g_1(x^*) = 0$, $g_2(x^*) > 0$, $g_3(x^*) = 0$, and $g_4(x^*) > 0$. Thus, $\omega_2^* = \omega_4^* = 0$ and

(i) $.10 - .078\, \omega_1^* - \omega_3^* = 0$

(ii) $.04 + .031\, \omega_1^* - \omega_3^* = 0.$

Hence, $\omega_1^* = .55$ and $\omega_3^* = .057$. It follows that $u_\gamma^* = -.55$, $u_{\alpha_1}^* = 55.6$, $u_\lambda^* = 0$, $u_{\alpha_2}^* = 0$, $u_{b_0^*}^* = -.057$, $u_{\alpha_3}^* = 10.4$, and $u_\delta^* = 0$.

Appendix A

Let Z be a normally distributed random variable with mean η and variance $\sigma^2 > 0$. Then,

(A1) $$P\{Z < t\} = \Phi((t - \eta)/\sigma)$$

where Φ is the cumulative standard normal distribution function. If $0 < \alpha < 1$, it follows that $P\{Z < t\} \leqq \alpha$ if, and only if,

(A2) $$(t - \eta)/\sigma \leqq \Phi^{-1}(\alpha).$$

This reduces to

(A3) $$t \leqq \eta + K(\alpha)\sigma$$

where $K = \Phi^{-1}$.

Suppose that $R = (R_1, \cdots, R_n)'$ is a random vector whose distribution is multivariate normal with mean vector μ and positive definite covariance matrix Γ. If x is any nonzero vector, then it can be shown that the random variable $R'x$ is normally distributed with mean $\mu'x$ and variance $x'\Gamma x$. From the previous discussion it follows that $P\{R'x < t\} \leqq \alpha$ if, and only if,

(A4) $$g(x) = \mu'x + K(\alpha)(x'\Gamma x)^{1/2} - t \geqq 0$$

when $x \neq 0$. The equivalence remains valid for the case $x = 0$.

For $\alpha < .5$, the function g in (A4) is concave, and it follows that the inequality induces a convex set. Geometrically, this induced set represents one nappe of a hyperboloid and its interior. The function g is differentiable everywhere except $x = 0$. Charnes and Cooper [7], have suggested that (A4) be replaced by

(A5) $$\begin{aligned} \mu'x + K(\alpha)v - t &\geqq 0 \\ -x'\Gamma x + v^2 &\geqq 0 \\ v &\geqq 0 \end{aligned}$$

for algorithmic solution. It is not difficult to verify (for $\alpha < .5$) that (A4) holds if, and only if, (A5) holds for some v. Moreover, (A4) is an equality if, and only if, the first two inequalities in (A5) are equalities. The constraints in (A5) are differentiable everywhere and are analytically more tractable.

Let R, Z be defined as above and uncorrelated. Let A be an $n \times n$ diagonal idempotent matrix, i.e., a diagonal element of A is either zero or one. Let c be an arbitrary n-dimensional vector. Since the random variable $(c + AR)'x + Z$ is normally distributed with mean $(c + A\mu)'x + \eta$ and variance $x'A\Gamma Ax + \sigma^2$, the inequality $P\{(c + AR)'x + Z < t\} \leqq \alpha$ is equivalent to

$$\text{(A6)} \qquad \tilde{g}(x) = (c + A\mu)'x + \eta + K(\alpha)(x'A\Gamma Ax + \sigma^2)^{1/2} - t \geqq 0$$

for all x. For $\alpha < .5$, the function $\tilde{g}$ in (A6) is concave and everywhere differentiable. When A is the zero matrix, the function is linear. It is clear that (A6) can be transformed into inequalities analogous to (A5).

Appendix B

Assuming that we have obtained the Kuhn Tucker multipliers or shadow prices, ω^*, the following formulas follow from equation (9) in the text. We employ the differentiation rule for inverse functions to obtain

$$K'(\alpha) = (\Phi^{-1})'(\alpha) = [\Phi'(\Phi^{-1}(\alpha))]^{-1} = [\varphi(K(\alpha))]^{-1},$$

where φ is the standard normal density function.

$$u_\gamma^* = \omega_1^*[\partial g_1(x^*)/\partial\gamma] = -\omega_1^*$$

$$u_{\alpha_1}^* = \omega_1^*[\partial g_1(x^*)/\partial\alpha_1] = \omega_1^*\sqrt{x^{*\prime}\Gamma x^*}/\varphi(K(\alpha_1))$$

$$u_\lambda^* = \omega_2^*[\partial g_2(x^*)/\partial\lambda] = -\omega_2^* p,$$

$$u_{\alpha_2}^* = \omega_2^*[\partial g_2(x^*)/\partial\alpha_2] = \omega_2^*\sqrt{x^{*\prime}\Gamma^* x^* + \tau^2}/\varphi(K(\alpha_2)),$$

$$u_{b_0^*}^* = \omega_3^*[\partial g_3(x^*)/\partial b_0^*] = -\omega_3^*,$$

$$u_{\alpha_3}^* = \begin{cases} \omega_3^*\tau/\varphi(K(\alpha_3)) & \text{if} \quad -\nu + K(\alpha_3)\tau < 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$u_\delta^* = \omega_4^*[\partial g_4(x^*)/\partial\delta] = \omega_4^* s.$$

References

1. Abadie, J. (ed.), *Nonlinear Programming*, Wiley, New York, 1967.
2. Baumol, W. J., "An Expected Gain-Confidence Limit Criterion for Portfolio Selection," *Management Science*, Vol. 10, (1963), pp. 174–182.
3. Charnes, A. and Cooper, W. W., *Management Models and Industrial Applications of Linear Programming*, Vol. I and II, Wiley, New York, 1961.
4. ——, —— and Symonds, G. H., "Cost Horizons and Certainty Equivalents: An Approach to Stochastic Programming of Heating Oil," *Management Science*, Vol. 4, (1958), pp. 235–263.
5. —— and ——, "Chance-Constrained Programming," *Management Science*, Vol. 6, (1959), pp. 73–79.
6. —— and ——, "Chance Constraints and Normal Deviates," *Journal of the American Statistical Association*, Vol. 57, (1962), pp. 134–148.
7. —— and ——, "Deterministic Equivalents for Optimizing and Satisficing Under Chance Constraints," *Operations Research*, Vol. II, (1963), pp. 18–39.
8. —— and Thore, S., "Planning for Liquidity in Financial Institutions: The Chance Constrained Method," *Journal of Finance* (1966), pp. 649–674.
9. Davis, S. C. *Weekly Underwriter*, April 26, 1952.
10. Denenberg, H. S. et al., *Risk and Insurance*, Prentice-Hall, Englewood Cliffs, N. J., 1965.
11. Gallagher, R. B., *Auditing the Corporate Insurance Function*, American Management Association Research Study 68, 1964.
12. Kuhn, K. W. and Tucker, A. W., "Nonlinear Programming," Proceedings of the Second Symposium on Mathematical Statistics and Probability, J. Neymans (ed), University of California Press, Berkeley, Calif., 1951, pp. 481–493.
13. Kulp, C. A., *Casualty Insurance* (3rd ed.), Ronald Press, New York, 1956.
14. Markowitz, H. M., "Portfolio Selection," *Journal of Finance*, Vol. 7, (1952), pp. 77–91.
15. Naslund, B. and Whinston, A., "A Model of Multi-Period Investments Under Uncertainty," *Management Science*, Vol. 8, (1962), pp. 184–200.