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# INEQUALITIES WITH APPLICATION IN ECONOMIC RISK ANALYSIS

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**Abstract**

Two sharp lower bounds for the expectation of a function of a non-negative random variable are obtained under rather weak hypotheses regarding the function, thus generalizing two sharp upper bounds obtained by Brook for the moment generating function. The application of these bounds to economic risk analysis is discussed.

TCHEBYCHEV INEQUALITIES; METHOD OF MARKOV; CONVEX FUNCTIONS; ECONOMIC RISK ANALYSIS; EXPECTED UTILITY; MEAN VARIANCE LOWER BOUNDS

## 1. Introduction

Let $X$ be a non-negative random variable with $\mu = E(X) > 0$ and $\lambda = E(X^2) < \infty$. In [3], Brook obtained sharp finite upper bounds, via the "Method of Markov" [8], for $E(\exp(\rho X))$ in terms of (i) $\mu$ and $\delta$, where $X \leqq \delta < \infty$, for arbitrary real $\rho$ and (ii) $\mu$ and $\lambda$ for $\rho \leqq 0$. In the next section, we generalize Brook's results in order to obtain sharp lower bounds for $E(f(X))$ under rather weak hypotheses regarding $f$. In the final section, we discuss briefly the application of these results to economic risk analysis.

We assume throughout that $f$ is a real-valued function on $[0,\infty)$ with $f(0) = 0$ and $g(x) = f(x)/x$ on $(0,\infty)$. We shall make use of some elementary properties of convex functions; the reader is referred to [5], [7], and [10].

## 2. The inequalities

*Theorem* 1. Suppose that $X \leqq \delta < \infty$ and that $g$ is non-increasing on $(0, \delta]$. Then,

$$E(f(X)) \geqq \mu g(\delta) = (\mu/\delta) f(\delta) \tag{1}$$

and the bound is sharp.

*Proof.* $E(f(X)) \geqq a\mu$ if $f(x) \geqq ax$ for all $x \in [0,\delta]$. $f(x) \geqq ax$ for all $x \in [0,\delta]$ if and only if $g(x) \geqq a$ for all $x \in (0,\delta]$ which is equivalent to $g(\delta) \geqq a$ since $g$ is non-increasing. Hence, the maximum possible value for $a$ is $g(\delta)$ and the resulting bound (1) is obviously attained when $X$ is concentrated at $\delta$.

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*Remarks.* It is not difficult to show (under our assumptions) that $f$ concave on $[0,\delta]$ implies $g$ non-increasing on $(0,\delta]$. If $f$ is continous on $(0,\delta]$ and differentiable on $(0,\delta)$, $g$ is non-increasing on $(0,\delta]$ if and only if $g' \leqq 0$ (or $f - xf' \geqq 0$) on $(0,\delta)$.

*Theorem* 2. Suppose that $g$ is convex on $(0,\infty)$, and put $\gamma = \lambda/\mu$. Then,

$$E(f(X)) \geqq \mu g(\gamma) = (\mu^2/\lambda) f(\lambda/\mu) \tag{2}$$

and the bound is sharp.

*Proof.* $E(f(X)) \geqq a\mu + b\lambda$ if $f(x) \geqq ax + bx^2$ for all $x \geqq 0$. $f(x) \geqq ax + bx^2$ for all $x \geqq 0$ if and only if $g(x) \geqq a + bx$ for all $x > 0$. Let $\xi > 0$ be arbitrary. By the convexity of $g$, there exists a $b(\xi)$ such that

$$g(x) \geqq g(\xi) + b(\xi)(x - \xi) \tag{3}$$

for all $x > 0$, and it follows that

$$E(f(X)) \geqq \mu(g(\xi) + b(\xi)(\gamma - \xi)) \tag{4}$$

for every $\xi > 0$. But (3) and (4) together imply that $\xi = \gamma$ yields the maximal bound, and the bound (2) is obviously attained when $X$ is concentrated at $\mu$.

*Remarks.* When both theorems apply, (2) always provides at least as tight a bound since $\gamma \leqq \delta$. If $f$ is twice differentiable on $(0,\infty)$, $g$ is convex on $(0,\infty)$ if and only if $g'' \geqq 0$ (or $2f - 2xf' + x^2 f'' \geqq 0$) on $(0,\infty)$.

*Examples.* The following three functions satisfy both Theorem 1 (for arbitrary $\delta$) and Theorem 2.

$$f(x) = 1 - \exp(-\rho x)\,; \quad \rho > 0 \tag{5}$$

$$f(x) = (x+\beta)^\alpha - \beta^\alpha; 0 < \alpha \leqq 1, \ \beta \geqq 0 \tag{6}$$

$$f(x) = \log(x + \beta) - \log\beta; \quad \beta > 0. \tag{7}$$

Inequalities (1) and (2) applied to (5) yield upper bounds for $E(\exp(-\rho X))$ that are equivalent to those of Brook [3] (Theorem 1 is satisfied for arbitrary real $\rho$). If $Y$ is a random variable with $Y \geqq \beta$, $E(Y) > \beta$, and $X = Y - \beta$, then the inequalities applied to (6) and (7) yield lower bounds for $E(Y^\alpha)$ and $E(\log Y)$ respectively (Theorem 2 also applies to (6) for $\alpha \geqq 2$). The inequalities applied to the composition of (5) and (6) yield upper bounds for $E(\exp(-\rho Y^\alpha))$, and they yield upper bounds for $E(Y^{-\rho})$ when applied to the composition of (5) and (7). Note that the functions (5), (6), and (7) are all of the form $f(x) = \int_0^x h(u)du$ where h is non-increasing, positive, and convex; referring to [4] and [10], our theorems apply to such functions under rather general conditions.

## 3. Application to economic risk analysis

Economic risk analysis is concerned with the following situation. An individual (group, firm, etc.) has a positive stock of wealth (capital) and a number of mutually exclusive alternative risky prospects (investments) for employing this wealth. The wealth associated with a risky prospect is assumed to be a non-negative random variable so that the present stock of wealth may be diminished, or even eliminated, but it cannot become negative. In this context, the individual's problem is to decide on a prospect. This decision problem has received a great deal of attention by economists; the reader is referred particularly to the books by Arrow [1] and Borch [2].

If the von Neumann-Morgenstern axioms [11] (or some version thereof) are accepted, a rational individual should construct a utility function (say by considering simple lotteries) and then select the prospect maximizing the expected utility of wealth. As an alternative to actually constructing a utility function, an individual can select one of the various functional forms deemed "reasonable" by economists [1], [2], [12] and specify the parameters (if any) to suit himself. We remark that versions of (5), (6), and (7) have all been suggested as reasonable utilities of wealth.

Now, even if the individual knows the distribution of wealth associated with each prospect, it may be difficult or impossible to compute all the expected utilities and to isolate the prospect maximizing expected utility. An alternative procedure is to compute lower bounds for the expected utilities and to isolate the prospect having the maximal lower bound. Moreover, this alternative is feasible (at least in principle) even when the various distributions are unknown, provided that lower bounds can be constructed. For at least some utility functions, our inequalities (1) and (2) provide a means of constructing lower bounds depending only on two (or perhaps three) distribution parameters. Economists may find this lower bound approach, when applicable, to be more useful (and palatable) than alternative approximate procedures.

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