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MAXIMIZATION OF LONG-RUN AVERAGE RATE-OF-RETURN BY STOCHASTIC APPROXIMATION

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ABSTRACT

Suppose that an individual has a surplus stock of wealth and a fixed set of risky investment opportunities over a sequence of time periods. Assuming the criterion of maximal long-run average rate-of-return, the individual may select portfolios sequentially via a modified stochastic approximation procedure. This approach yields optimal asymptotic investment results under minimal assumptions.

1. INTRODUCTION

Suppose that an individual has a stock of wealth which is not required for short-run consumption expenditures and a fixed set of risky investment opportunities over an indefinite horizon of discrete time periods. Assuming time homogeneity and period-wise independence (or near independence) of investment results, Breiman [6], [7] has demonstrated that a portfolio (i.e., an allocation of wealth to the various investment opportunities) which maximizes the expected logarithm of period-wise growth is "optimal" in the sense that it maximizes the long-run average rate-of-return (or rate of growth) as the time horizon tends to infinity. (Actually, the treatment in [6] is somewhat more general.) Moreover, Breiman's optimal portfolio is a "fixed proportions" portfolio in that the proportions of wealth allocated to the various investment opportunities do not vary with the magnitude of wealth.

Now, if the individual accepts Breiman's optimality criterion, he will want to obtain the optimal proportions and employ them at every future period. However, in any realistic situation, the individual can only estimate the optimal proportions initially, and he will want to improve his estimates (and hence his investment results) as he goes along. A good sequential approximation scheme should yield the optimal proportions and should attain the maximal average rate-of-return in the limit as the time horizon tends to infinity. (It appears that even the best conceivable statistical procedures will have sufficient asymptotic variation to render them "inadmissible" in the sense of Breiman [6]; i.e., the individual would be infinitely better off if he knew the optimal proportions. Of course, our purpose is to treat the case where these optimal proportions are unknown.)

As a suitable approximation scheme, we suggest a modified (generally multivariate) Kiefer-Wolfowitz stochastic approximation procedure [24], [10] to [15], [32], [33], [37], [38]. This approach requires minimal assumptions and calculations, and it yields the desired results in the limit. After some preliminaries in the next section, the approximation procedure is presented in section 3 and its convergence properties are discussed. A simulated numerical example is presented in section 4, and section 5 discusses the addition of withdrawals for consumption.

2. PRELIMINARIES

Let $G_1, \ldots, G_k$ be nonnegative random variables representing the "growth factors" associated with $k \geq 1$ risky investment opportunities over a given time period; i.e., a dollar invested in opportunity j yields G_j dollars at the end of the period. We assume that $E(G_j^2) < \infty$ for all $j = 1, \ldots, k$.

Let x_j be the proportion of wealth invested in opportunity j. We shall require that $\sum_{j=1}^{k} x_j \leq \gamma$ for some $\gamma \epsilon (0, 1)$; $1 - \sum_{j=1}^{k} x_j \geq 1 - \gamma > 0$ then represents the proportion of wealth held in a "riskless asset" assumed to have growth factor $g \geq 1$ with probability one. Let

$$S = \left\{ x = (x_1, \ldots, x_k) : \sum_{j=1}^{k} x_j \leq \gamma; \qquad x_1, \ldots, x_k \geq 0 \right\} \tag{1}$$

represent the set of feasible proportional allocations or "portfolios." Given $x \epsilon S$, let

$$G(x) = g\left(1 - \sum_{j=1}^{k} x_j\right) + \sum_{j=1}^{k} G_j x_j = g + \sum_{j=1}^{k} (G_j - g) x_j \tag{2}$$

represent the portfolio growth factor over the given period; i.e., a dollar invested in portfolio $x \epsilon S$ yields $G(x)$ dollars at the end of the period. According to Breiman's criterion, the objective is to maximize

$$f(x) = E(\log (G(x))) \tag{3}$$

over S. This expectation exists because $G(x) \geq g(1-\gamma) > 0$ and $E(G(x)) < \infty$. In fact, it is easy to see that $E(|\log (G(x))|^p)$ is uniformly bounded on S for any integer $p \geq 1$.

By the strict concavity of the log function, f is concave on S. Moreover, it is strictly concave if the $G_j - g$ are linearly independent (i.e., $P\left\{\sum_{j=1}^{k} (G_j - g)\xi_j = 0\right\} < 1$ whenever $\xi = (\xi_1, \ldots, \xi_k) \neq 0$), and this latter condition holds if the covariance matrix of the G_j is positive definite. We shall assume strict concavity. Furthermore, we shall assume that f attains its unique maximum at an interior point $\theta \epsilon S$; i.e., none of our investment opportunities, including the riskless asset, is completely inferior to any other, and the optimal portfolio is "diversified." Since the riskless portfolio is feasible, it follows that $f(\theta) > \log (g) \geq 0$.

Under our previous assumptions, the first and second order partial derivatives of f exist and are uniformly bounded in absolute value on S. Indeed, we have

$$\partial f / \partial x_j (x) = E((G_j - g)/G(x)) \tag{4}$$

so the optimality condition $\partial f / \partial x_j (\theta) = 0$ implies

$$E(G_j / G(\theta)) = E(g / G(\theta)) = 1 \tag{5}$$

which indicates that the optimal portfolio "balances" the various investment opportunities in an intuitive way. The second order derivatives

(6) $$\partial^2 f/\partial x_i \partial x_j(x) = -E((G_i - g)(G_j - g)/G(x)^2)$$

need not be continuous, but continuity would follow from the existence of higher moments for the G_j. In general, the existence of pth order moments for the G_j implies the existence and boundedness of pth order derivatives for f. If, in particular, the G_j are bounded random variables, derivatives of all orders exist on S.

3. THE APPROXIMATION PROCEDURE

Let $W_0 > 0$ be our initial wealth, and let $X_1 = (X_{1,1}, \ldots, X_{1,k})$ be our initial estimate of θ. We assume that $c < \min_j X_{1,j}$ and $\sum_{j=1}^{k} X_{1,j} < \gamma - c$, where $0 < c < \gamma/(k+1)$. Define positive sequences $\{a_n\}$ and $\{c_n\}$ by $a_n = an^{-\alpha}$ and $c_n = cn^{-\beta}$, where $0 < \beta < 1/2$ and max $(1/2+\beta, 1-2\beta) < \alpha < 1$. Let e_j denote the jth unit vector of dimension k, and let $h(u) = 0$ for $u \leq 0$, $h(u) = u$ for $u > 0$.

Our approximation procedure will evolve over a sequence of "stages," each stage encompassing $2k$ consecutive time periods. Our estimate of θ at the nth stage will be denoted by $X_n = (X_{n,1}, \ldots, X_{n,k})$, and our wealth after the nth stage will be denoted by W_n. During the nth stage, we implement the $2k$ portfolios $X_n \pm c_n e_j$ $(j = 1, \ldots, k)$, and we realize the corresponding growth factors

(7) $$G^{\pm}_{n,j} \stackrel{d}{=} G(X_n \pm c_n e_j),$$

where $\stackrel{d}{=}$ means "identically distributed," and $G(x)$ denotes the random function defined in section 2. It follows then that

(8) $$E(|\log(G^{\pm}_{n,j})|^2) \leq C < \infty$$

and assuming period-wise independence (or near independence) of component growth factors

(9) $$E(\log(G^{\pm}_{n,j})|X_n, \ldots, X_1) = f(X_n \pm c_n e_j)$$

so the Kiefer-Wolfowitz scheme would ordinarily put

(10) $$X_{n+1} = X_n + a_n Y_n,$$

where $Y_n = (Y_{n,1}, \ldots, Y_{n,k})$, and

(11) $$Y_{n,j} = (2c_n)^{-1}(\log(G^{+}_{n,j}) - \log(G^{-}_{n,j}))$$

is an estimate of $\partial f/\partial x_j(X_n)$. However, we must modify the basic recursion relation (10) in order to ensure feasibility. Our proposed modification is in no sense unique, but it is rather simple.

We put

$$J_n = \{j : Y_{n,j} > 0\}, \tag{12}$$

$$A_{n,j} = Y_{n,j} \Big/ \sum_{i \in J_n} Y_{n,i} \qquad (j \in J_n), \tag{13}$$

$$B_{n,j} = \max\,(c_n, X_{n,j} + a_n Y_{n,j}), \tag{14}$$

$$X_{n+1,j} = B_{n,j} > c_{n+1} \qquad (j \notin J_n), \tag{15}$$

$$X_{n+1,j} = B_{n,j} - A_{n,j} h \left(\sum_{i=1}^{k} B_{n,i} - \gamma + c_n \right) \qquad (j \in J_n), \tag{16}$$

and it is not difficult to establish

$$X_{n+1,j} > X_{n,j} \qquad (j \in J_n), \tag{17}$$

$$\sum_{j=1}^{k} X_{n+1,j} \leq \gamma - c_n < \gamma - c_{n+1}, \tag{18}$$

from which it follows that all implemented portfolios $X_n \pm c_n e_j$ are interior points of the feasible set S.

Since θ is an interior point of S by assumption, the asymptotic behavior of the approximation procedure will be unaffected by the feasibility requirements. Under our assumptions (cf. Theorem 4.4 in [33]), $X_n \to \theta$ with probability one as $n \to \infty$. Moreover,

$$\log(W_n) - \log(W_{n-1}) = \sum_{j=1}^{k} (\log(G^+_{n,j}) + \log(G^-_{n,j})) \tag{19}$$

so that $n^{-1}(\log(W_n) - \log(W_0)) \to 2kf(\theta)$ or $W_n^{1/n} \to \exp(2kf(\theta))$ with probability one by the Strong Law for Martingales [18]. Hence, the optimal proportions and the maximal average'rate-of-return are attained almost surely in the limit as the time horizon tends to infinity.

Now, a, c, α, β, γ, and the vector X_1 are parameters to be stipulated by the individual. Obviously, X_1 should be the individual's best initial estimate of θ, and γ should be chosen sufficiently close to one so that θ is certain to be an interior point of the feasible set S. Furthermore, it is clear that a should be sufficiently small relative to c so that the procedure will avoid the boundary at the first iteration. (Indeed, a can be stipulated after the first stage results are known.)

Under slightly stronger assumptions, we can investigate the effect of various choices of a, c, α, and β on the asymptotic behavior of our approximation procedure. Suppose, for instance, that our basic growth factors in section 2 possess finite third moments so that f has bounded third derivatives on S. In addition, suppose that the minimum eigenvalue of $-H(x)$ is bounded away from zero on S, where

(20) $$H(x) = [\partial^2 f / \partial x_i \partial x_j (x)]_{i,j=1}^{k}$$

is the Hessian of f. Under these assumptions, $0 < \beta < 1/6$ and $\alpha = 6\beta$ yield

(21) $$E(||X_n - \theta||^2) = 0(n^{-4\beta})$$

(cf. Theorem 4.6 in [33]) so the procedure converges in the mean square even if the various conditions on α and β for probability one convergence are violated. Indeed, for β very close to zero and $\alpha = 6\beta$, our procedure is akin to a perpetual control process.

Assuming further that f has continuous third derivatives and that $\alpha = 6\beta$ with $1/8 < \beta < 1/6$, we have $X_n \to \theta$ with probability one, and in addition $n^{2\beta}(X_n - \theta)$ is asymptotically normal (cf. Theorem 4.7 in [33]) with mean vector

(22) $$\mu = c^2 m$$

and covariance matrix

(23) $$V = (a/c^2)\sigma^2 M,$$

where

(24) $$\sigma^2 = \operatorname{var}(\log(G(\theta)))$$

and m is a vector, M a matrix, dependent only on the behavior of f in the immediate vicinity of θ. It follows that asymptotically

(25) $$E(||X_n - \theta||^2) \simeq n^{-4\beta}(b_1 c^4 + b_2(a/c^2)),$$

where b_1 and b_2 are positive constants which are independent of the choice of parameters and which are in some sense inversely proportional to the degree of curvature in f at θ. From this result, it is clear that asymptotic speed and precision are increased as β is increased, as c is decreased, and as a is decreased relative to c. On the other hand, if a is chosen too small, the procedure will barely move at all in the short run, and this may be undesirable if X_1 is not a particularly good estimate of θ. In general, the individual must stipulate the parameters a, d, and β subjectively in order to balance his short and long term expectations.

4. SIMULATED NUMERICAL EXAMPLE

Consider a simplified environment where cash is the riskless asset with $g = 1$ and the single risky investment opportunity is a simple "double or nothing" favorable gamble (cf. [19], [35], [36]) with constant probability success $p = 0.55$. Then,

(26) $$f(x) = p \log(1 + x) + q \log(1 - x)$$

with $q = 1 - p = 0.45$. Furthermore,

(27) $$\theta = p - q = 0.10$$

(28) $$\exp(f(\theta)) = 2p^p q^q \simeq 1.005$$

(29) $$f''(\theta) = -1/4pq \simeq -1.010$$

(30) $$f'''(\theta) = -\theta/4(pq)^2 \simeq -0.408$$

(31) $$\sigma^2 = pq(\log(p/q))^2 \simeq 0.010$$

and the constants in (25) are

(32) $$b_1 = (f'''(\theta)/6f''(\theta))^2 \simeq 0.0045$$

(33) $$b_2 = \sigma^2/4|f''(\theta)| \simeq 0.0025.$$

Now suppose that our individual has 50 independent gambling opportunities per year so that the optimum annualized long-run rate-of-return is about 28.4 percent. Since $k=1$, 50 trials correspond to 25 stages per year in our approximation procedure, and we may simplify the notation of section 3 to

(34) $$G_n^{\pm} = 1 + (2U_n^{\pm} - 1)(X_n \pm c_n)$$

(35) $$Y_n = (2c_n)^{-1}(\log(G_n^+) - \log(G_n^-))$$

(36) $$X_{n+1} = \max(c_n, \min(\gamma - c_n, X_n + a_n Y_n))$$

(37) $$W_n = \exp\left(\sum_{i=1}^{n}(\log(G_i^+) + \log(G_i^-))\right)$$

where $X_n \pm c_n \epsilon (0, \gamma)$ are simply the proportions of wealth gambled at the two trials in stage n, $U_n^{\pm}$ are the corresponding zero/one (lose/win) indicator random variables, and $W_0 \equiv 1$ for simplicity.

In order to better illustrate how our approximation procedure works, we have simulated 10 1-year runs with $X_1 = 0.08$, $c = 0.05$, $a = 0.001$ and 0.01, $\gamma = 0.20$, $\beta = 0.15$, and $\alpha = 6\beta = 0.90$. The random numbers employed were taken from the first 10 columns on p. 480 of [4]; i.e., each column of 50 random numbers was used to generate a 1-year simulation run. The net results are listed in Table 1, whereas Run 2 is detailed step-by-step in Table 2. Note the difference in short-run variation between the cases $a = 0.001$ and $a = 0.01$.

5. CONCLUDING REMARKS

Suppose that our individual wishes to make withdrawals from his stock of wealth for purposes of consumption. Joint optimization of consumption-investment decisions over time has generally been

TABLE 1. *Net Simulation Results*

Run	$a = 0.001$		$a = 0.01$	
	X_{26}	W_{25}	X_{26}	W_{25}
1	0.0825	2.997	0.1044	3.785
2	0.0816	1.364	0.0958	1.334
3	0.0789	1.281	0.0679	1.229
4	0.0817	1.288	0.0966	1.220
5	0.0776	2.208	0.0602	1.698
6	0.0788	3.612	0.0694	2.779
7	0.0764	0.409	0.0486	0.480
8	0.0817	1.124	0.0972	1.047
9	0.0820	1.715	0.0993	1.691
10	0.0818	1.693	0.0970	1.735
Average	0.0803	1.769	0.0836	1.700

TABLE 2. *Simulation Run 2*

n	U_n^+	U_n^-	$a = 0.001$		$a = 0.01$	
			X_n	W_n	X_n	W_n
1	1	1	0.0800	1.164	0.0800	1.164
2	1	0	0.0809	1.264	0.0893	1.262
3	1	1	0.0818	1.477	0.0994	1.523
4	0	0	0.0822	1.241	0.1028	1.223
5	1	1	0.0819	1.451	0.0996	1.477
6	0	1	0.0821	1.333	0.1018	1.351
7	0	0	0.0817	1.122	0.0962	1.102
8	0	1	0.0815	1.034	0.0943	1.013
9	1	1	0.0811	1.207	0.0902	1.203
10	1	0	0.0812	1.286	0.0914	1.279
11	0	0	0.0815	1.083	0.0946	1.047
12	1	1	0.0814	1.266	0.0933	1.250
13	1	0	0.0815	1.345	0.0943	1.326
14	1	0	0.0817	1.428	0.0970	1.404
15	1	1	0.0819	1.670	0.0996	1.696
16	1	0	0.0820	1.771	0.1004	1.793
17	0	0	0.0822	1.490	0.1028	1.441
18	0	1	0.0821	1.384	0.1019	1.334
19	0	1	0.0819	1.288	0.0995	1.237
20	0	0	0.0817	1.084	0.0972	1.007
21	1	0	0.0817	1.147	0.0965	1.062
22	1	1	0.0818	1.341	0.0984	1.280
23	1	1	0.0819	1.569	0.0990	1.545
24	0	1	0.0819	1.462	0.0995	1.435
25	0	1	0.0818	1.364	0.0976	1.334
26			0.0816		0.0958	

addressed in the literature via two criteria: maximum discounted expected utility of long-run consumption (or a suitable combination of consumption and wealth) [17], [21], [22], [25], [27], [28], [30] and minimum probability of ruin in the context of rigid consumption requirements [19], [35], [36].

Both criteria yield rather complicated dynamic programming type functional equations to be solved for the optimum consumption-investment policies. (It should be mentioned, however, that a variety of intriguing inferential results have been obtained via these characterizations.) We shall ignore the problem of joint optimization and merely indicate an intuitive means of adding a flexible consumption policy to the approximation procedure developed in section 3.

Let ρ_n be a nonnegative sequence converging to zero (e.g., $\rho_n \equiv 0$). Referring to section 3, we put

$$Z_n = \sum_{j=1}^{k} (\log (G_{n,j}^+) + \log (G_{n,j}^-)) \tag{38}$$

$$V_n = n^{-1} \sum_{i=1}^{n} Z_i = n^{-1} Z_n + (1 - n^{-1}) V_{n-1} \tag{39}$$

($V_0 = 0$) so that $V_n \to 2kf(\theta)$ with probability one. Putting

$$T_n = \max (\rho_n, V_n), \tag{40}$$

we also have $T_n \to 2kf(\theta)$ with probability one. Now let C_n be the withdrawal for consumption at the end of stage n and let W_n be the postwithdrawal wealth so that

$$W_n + C_n = W_{n-1} \exp (Z_n). \tag{41}$$

Our policy is

$$W_n = (W_n + C_n) \exp (-\delta T_n) \tag{42}$$

$$C_n = (W_n + C_n)(1 - \exp (-\delta T_n)), \tag{43}$$

where $\delta \epsilon (0, 1)$ is a subjective parameter. (The procedure in section 3 corresponds to $\delta = 0$.)

We have then that $W_n^{1/n}$ and also $C_n^{1/n}$ converge to $\exp ((1-\delta) 2kf(\theta))$ with probability one so that δ provides a flexible mechanism for trading off short and long term consumption. The long term is weighted heavily when δ is near zero, whereas the short term is weighted heavily when δ is near one. (Note that our stage-wise withdrawal policy could easily be made period-wise.)

Given an extensive set of risky investment opportunities, it may be necessary to aggregate in order to render the approximation problem manageable. For instance, one can imagine a one-dimensional trade-off between a riskless asset and a suitably defined "market portfolio" [23].

Regarding speed of convergence, we note that asymptotic speed can be improved via Fabian's generalized Kiefer-Wolfowitz procedures at the expense of increased analytical complexity [13], [14], [15], [33]. Nevertheless, any statistical procedure will converge rather slowly in comparison to deterministic procedures. On the other hand, deterministic procedures do not apply directly to problems where randomness is a fundamental ingredient. Any rigorous statistical procedure will proceed rather cautiously in order to properly screen valid information about the underlying objective function from pure random noise [38].

Samuelson [31] has questioned the long-run average rate-of-return criterion on the basis of alternative, finite-stage utility criteria. He notes that not all individuals need be limiting logarithmic utility maximizers. While this is undoubtedly true, logarithmic utility is distinguished by the fact that it does possess some external justification. On the other hand, one can easily tailor our approximation procedure to a modified criterion such as

$$f(x) = E(G(x)^{\eta}) \tag{44}$$

$(0 < \eta < 1)$ which would maximize $E((W_n/W_{n-1})^{\eta})$ in the limit. In any event, the infinite stage setup is obviously convenient when the ultimate number of stages is undetermined.

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