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INEQUALITIES WITH APPLICATION IN RETAIL INVENTORY ANALYSIS

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Abstract

Simple bounds on service level and turnover velocity are obtained for a periodic-review inventory system with a stationary order-up-to-level stocking policy and no backordering. Exact computational formulas are given for Poisson demand. An illustrative numerical example is presented, and the application of these bounds to retail inventory analysis is discussed.

SERVICE LEVEL; TURNOVER VELOCITY; RETAIL INVENTORY ANALYSIS; POISSON DEMAND

1. Introduction

Consider a periodic-review inventory system where stock is reviewed every ω weeks and the stationary stocking rule is simply 'order up to k units' including on-hand plus on-order (assuming flexible order quantities). Suppose that the delivery lead time is ν weeks, that the demand process is stationary and simple (i.e., one unit per customer), and that unserviced demand is permanently lost (i.e., no backordering). Under these conditions, there will obviously exist steady-state (or long-run) values $\rho(k)$ for 'service level' (i.e., fraction of demand serviced) and $\tau(k)$ for 'turnover velocity' (i.e., annual sales over time-averaged inventory), but they are difficult to compute exactly, even under the assumption of Poisson demand [6]. On the other hand, it is relatively easy to compute bounds for $\rho(k)$ and $\tau(k)$.

In the next section, we derive inequalities for service level and turnover velocity, with simple computational formulas in the case of Poisson demand. The subsequent sections contain an illustrative numerical example and some further discussion of the application of our bounds in retail inventory analysis.

Our demand process $N(t)$ is assumed to be a cumulative, non-compound counting process on $(0,\infty)$ with stationary increments and $E(N(t)) = \lambda t$ for some $\lambda > 0$. We take the time origin at a review epoch, and time is scaled in weeks so that λ is scaled in units per week. We employ the Poisson notation $p(j;\mu) = \exp(-\mu)\mu^j/j!$ and $P(j;\mu) = \Sigma_{i=0}^{j} p(i;\mu)$. Also, we use the notation $x \wedge y$ for the minimum of x and $y, x \vee y$ for the maximum of x and $y, x^+ = x \vee 0$, and the

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identity $x \wedge y = x - (x-y)^+$. Finally, we refer to k throughout as the OUTL (order-up-to-level).

2. The inequalities

Theorem 1. If the OUTL is $k \geqq 1$ units, then the expected unit sales (i.e., serviced demand) in the time interval $(\nu, \nu+\omega]$ is at least

$$S(k) = E(k \wedge N(\nu+\omega)) - E(k \wedge N(\nu)). \tag{1}$$

Proof. Let $I(t)$ denote the on-hand inventory at epoch t. If the OUTL is k, then $I(\nu) \geqq (k - N(\nu))^+$ since all outstanding orders will have arrived and at most $N(\nu)$ units will have been sold in the time interval $(0, \nu]$. Hence, the expected sales

$$\begin{aligned} &E(I(\nu) \wedge (N(\nu+\omega) - N(\nu))) \\ &\quad \geqq E((k - N(\nu))^+ \wedge (N(\nu+\omega) - N(\nu))) \\ &\quad = E((k-N(\nu))^+) - E(((k - N(\nu))^+ - (N(\nu+\omega) - N(\nu)))^+) \\ &\quad = E((k-N(\nu))^+) - E((k - N(\nu+\omega))^+) \\ &\quad = E(k \wedge N(\nu+\omega)) - E(k \wedge N(\nu)). \end{aligned}$$

Corollary 2. If the OUTL is $k \geqq 1$ units, then the steady-state (i.e., long-run) service level

$$\rho(k) \geqq \alpha(k) = S(k)/\lambda\omega. \tag{2}$$

Proof. By stationarity, the long-run average sales per ω weeks is at least $S(k)$, and the long-run average demand per ω weeks is $\lambda\omega$. Hence, the long-run fraction of demand serviced is at least $\alpha(k)$.

Theorem 3. If the OUTL is $k \geqq 1$ units, then the expected unit average inventory in the time interval $(\nu, \nu+\omega]$ is at least

$$\begin{aligned} A(k) &= \int_\nu^{\nu+\omega} E((k - N(t))^+) dt/\omega \\ &= k - \int_\nu^{\nu+\omega} E(k \wedge N(t))\, dt/\omega. \end{aligned} \tag{3}$$

Proof. Let $I(t)$ denote the on-hand inventory at epoch t. If the OUTL is k, then $I(t) \geqq (k - N(t))^+$ for all $t \in (\nu, \nu+\omega]$ since all outstanding orders will have arrived and at most $N(t)$ units will have been sold in the time interval $(0,t]$. Hence, the expected average inventory

$$E\left(\int_{\nu}^{\nu+\omega} I(t)\,dt/\omega\right) = \int_{\nu}^{\nu+\omega} E(I(t))\,dt/\omega$$

(cf. Prabhu [8], p. 28) and the assertion follows by simple dominance.

Corollary 4. If the OUTL is $k \geqq 1$ units, then the steady-state (i.e., long-run) turnover velocity

$$\tau(k) \leqq \beta(k) = 52\lambda / A(k). \tag{4}$$

Proof. By stationarity, the long-run average sales is no greater than 52λ units per year, and the long-run time-averaged inventory is at least $A(k)$. Hence, the long-run turnover velocity is no greater than $\beta(k)$.

Remark 5. By Jensen's Inequality [5], we have

$$A(k) \geqq \int_{\nu}^{\nu+\omega} (k - \lambda t)^{+} dt/\omega = k - \int_{\nu}^{\nu+\omega} (k \wedge \lambda t) dt/\omega$$

and the right side of this inequality is recognisable as the old 'saw-tooth curve' approximation to average inventory. Hence, the lower bound $A(k)$ is always at least as tight as the lower bound resulting from the saw-tooth curve.

Theorem 6. If $N(t)$ is a Poisson process, then

$$S(k) = \sum_{j=0}^{k-1} (P(j;\lambda\nu) - P(j;\lambda(\nu+\omega))) \tag{5}$$

$$A(k) = \sum_{j=0}^{k-1} \sum_{i=0}^{j} (P(i;\lambda\nu) - P(i;\lambda(\nu+\omega)))/\lambda\omega \tag{6}$$

for $k \geqq 1$.

Proof. It is not difficult to show that

$$\begin{aligned} E(k \wedge N(t)) &= \lambda t P(k-1;\lambda t) + k(1 - P(k;\lambda t)) \\ &= \sum_{j=0}^{k-1} (1 - P(j;\lambda t)) \end{aligned}$$

from which (5) follows immediately. Now,

$$\begin{aligned} \int_{\nu}^{\nu+\omega} E(k \wedge N(t))dt &= \sum_{j=0}^{k-1} \int_{\nu}^{\nu+\omega} (1 - P(j;\lambda t))dt \\ &= k\omega - \sum_{j=0}^{k-1} \sum_{i=0}^{j} \int_{\lambda\nu}^{\lambda(\nu+\omega)} p(i;u)du/\lambda \end{aligned}$$

but $\int_0^v p(i;u)\,du = 1 - P(i;v)$ and (6) follows by substitution. We note that (5) and (6) may also be verified via the analogous formulas in Appendix A3 of [6].

Remark 7. A very simple recursive computational algorithm arises naturally from Equations (5) and (6). Let

$$P_1(0) = p_1(0) = \exp(-\lambda\nu)$$
$$P_2(0) = p_2(0) = \exp(-\lambda(\nu+\omega))$$
$$S(0) = A(0) = 0$$

and for $k = 1, 2, \cdots$

$$p_1(k) = (\lambda\nu/k)p_1(k-1)$$
$$P_1(k) = P_1(k-1) + p_1(k)$$
$$p_2(k) = (\lambda(\nu+\omega)/k)p_2(k-1)$$
$$P_2(k) = P_2(k-1) + p_2(k)$$
$$S(k) = S(k-1) + P_1(k-1) - P_2(k-1)$$
$$\alpha(k) = S(k)/\lambda\omega$$
$$A(k) = A(k-1) + \alpha(k)$$
$$\beta(k) = 52\lambda/A(k).$$

This algorithm is viable for $\lambda(\nu+\omega) \leqq 50$. Beyond that, a Gaussian approximation is efficacious; i.e., assume that $N(t)$ is a cumulative Gaussian process with $\operatorname{var}(N(t)) = \lambda t$ and derive computational formulas for $S(k)$ and $A(k)$. The Gaussian formulas can be obtained from results in Appendix A4 of [6], although the expression for $A(k)$ seems rather complicated; an alternative, of course, is the saw-tooth curve bound.

Remark 8. A natural question is: How good are the bounds $S(k)$ and $A(k)$, and hence $\alpha(k)$ and $\beta(k)$? It follows from the very nature of our approximation to on-hand inventory that: (i) all these bounds (even the saw-tooth bound) become exact as $k \to \infty$ and (ii) the bounds $S(k)$ and $A(k)$ become exact as $\nu \to 0$ for fixed k. To obtain precise error estimates, one must resort either to exact Markovian analysis or to direct simulation.

The Markovian method involves calculation of the stationary (or invariant) distribution of the quantity on-hand immediately following a delivery (call it $\{\pi(j;k): j \geqq 0\}$ corresponding to stationary OUTL $k \geqq 1$) via the usual Markov chain approach ([4], [5], [8], [9]). (The 'state' of the Markov chain may involve on-order quantities as well.)

Exact formulas are then

$$\bar{S}(k) = \sum_j \pi(j;k)S_0(j)$$

$$\bar{A}(k) = \sum_j \pi(j;k)A_0(j)$$

$$\rho(k) = \bar{S}(k)/\lambda\omega$$

$$\tau(k) = 52\lambda\rho(k)/\bar{A}(k)$$

where S_0 and A_0 denote the exact results for $\nu = 0$. The difficulty with exact Markovian analysis, of course, is the calculation of the stationary distribution $\{\pi(j;k): j \geqq 0\}$; this calculation is generally cumbersome, but it is reasonably expeditious in certain special cases (e.g., $\nu = \omega$).

The author has tested the bounds derived from (5) and (6) via direct simulation, but the detailed results will not be reported here. In general, our simulation results confirm the relative sharpness of these bounds in the OUTL range $k \geqq \lambda(\nu + \omega) + 1.5(\lambda(\nu + \omega))^{1/2}$. In this paper, we have chosen to work out a single numerical example in detail, including exact comparative values for service level and turnover velocity.

3. A numerical example

Suppose that $\lambda = 0.5$ units per week (or 26 units per year), $\omega = 4$ weeks, and $\nu = 4$ weeks, a fairly typical set of parameters for an individual item in a retail store. Assuming Poisson demand, the following table was computed via the algorithm in Remark 7 and the Markovian method outlined in Remark 8.

TABLE I

k	$\rho(k)$	$\alpha(k)$	$\tau(k)$	$\beta(k)$	$\alpha(k)\beta(k)$
5	0.865	0.806	9.6	12.0	9.7
6	0.930	0.905	7.6	8.5	7.7
7	0.967	0.958	6.2	6.5	6.2
8	0.986	0.983	5.1	5.2	5.1
9	0.995	0.994	4.3	4.3	4.3
10	0.998	0.998	3.7	3.7	3.7

We have only tabulated values for $k > \lambda(\nu + \omega)$, the usual interval of interest. In the range $k \geqq 7$, our bounds $\alpha(k)$ and $\beta(k)$ provide reasonable approximations to the actual values of service level and turnover velocity, $\rho(k)$ and $\tau(k)$; moreover, the product $\alpha(k)\beta(k)$ essentially coincides with $\tau(k)$.

4. Application to retail inventory analysis

The author has found the bounds on service level and turnover velocity to be very useful in performing inexpensive service-turnover trade-off analyses and parameter sensitivity analyses for 'staple' retail store items. The Palm-Khintchine superposition theorem and its extensions [7], [1], [2], [3] lead one to expect Poisson customer arrivals (even when demand is non-stationary). Moreover, many retail items are sold one at a time, and steady-state results make sense for chains (i.e., 'ensembles' [6]) of retail stores, even over relatively short time horizons. The author has also found the following type of approximate marginal economic analysis to be useful in merchandise profit planning, assuming that $\alpha(k)$ and $A(k)$ are reasonable approximations to the corresponding actual values of service level and average inventory (at least for $\alpha(k)$ near one where the bounds are sufficiently sharp).

Assuming a unit acquisition cost of c, a unit selling price of $p > c$, and an annual inventory carrying rate of ξ, one can construct an approximate average annual profit function

$$\Pi(k) = (p - c)(52\lambda\alpha(k)) - \xi c A(k) \tag{7}$$

and perform marginal analysis to determine an optimum OUTL k^* (or an optimum service-turnover combination) for a fixed set of parameters λ, ω and ν. We ignore here price-demand elasticity and the cost of reviewing (and ordering) stock, which impacts ω (usually over an aggregate of items). Obviously,

$$\begin{aligned} k^* &= \max\{k \geqq 0: \Delta\Pi(k) \geqq 0\} \\ &= \max\{k \geqq 0: (p/c - 1)(52\lambda\,\Delta\alpha(k)/\Delta A(k)) \geqq \xi\} \end{aligned} \tag{8}$$

where Δ is the usual difference operator (e.g., $\Delta\Pi(k) = \Pi(k) - \Pi(k-1)$) and $k^* = 0$ implies that the item should not be carried in stock. In words, (8) simply says that the OUTL k is increased from zero so long as the marginal rate of return on inventory investment exceeds the marginal cost. The term $52\lambda\,\Delta\alpha(k)/\Delta A(k)$ approximates the 'marginal' turnover velocity of the inventory *added* when the OUTL is increased from $k-1$ to $k, p/c - 1$ is the markup rate, and the product approximates marginal rate of return.

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ADDENDUM

As indicated in Hadley & Whitin, the lost-sales (no backordering) model is quite complex, even for the simple periodic-review, order-up-to-level system treated in this paper. On the other hand, our lower bound estimates for sales, service level, and average inventory are quite simple, and they are very good in the high service level domain.

The economic trade-off logic embedded in equation 8, with Poisson demand in units of "average purchase amount," was the basis for an assortment planning system at Montgomery Ward in the 1970s. It was used to prototype ordering policy and average inventory for a broad array of staple retail store items, everything from men's underwear to shock absorbers. When prototype demand for an item was sufficiently low, and the economics didn't support any inventory, it was trimmed from the assortment. This assured that assortments were tailored a priori to different store environments, leading to lower inventory investment and fewer markdowns. Obviously, this and other OR innovations didn't save the company from its ultimate demise, but they did improve results at the time.