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SEQUENTIAL BID SELECTION BY STOCHASTIC APPROXIMATION

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ABSTRACT

Suppose that a contractor is faced with a sequence of "minimum bid wins contract" competitions. Assuming that a contractor knows his cost to fulfill the contract at each competition and that competitors are merely informed whether or not they have won, bids may be selected sequentially via a tailored stochastic approximation procedure. The efficacy of this approach in certain bidding environments is investigated.

1. INTRODUCTION

Consider the sequential decision problem of a contractor facing an interminable sequence of "minimum bid wins contract" (sealed-bid) competitions for similar contracts. If a random bidding environment is assumed where the competitions are independent and the probability of winning any competition is merely a function of percentage markup over cost, the contractor will naturally attempt to approximate a suitable percentage markup via some sequential approximation procedure that incorporates the data available from past competitions. The form of the data available depends on the information divulged to the competitors after each competition. Under minimal disclosure, a competitor is merely informed whether or not he has won.

In this paper, we suggest a Kiefer-Wolfowitz type stochastic approximation procedure for the purpose of converging on that percentage markup which maximizes expected percentage profit. The procedure assumes only minimal disclosure, and it converges under relatively weak assumptions regarding the p function, i.e., the probability of winning as a function of percentage markup. In the next section, we delineate those assumptions. In Section 3, the approximation procedure is presented and its convergence properties are discussed. Section 4 and the Appendix contain the results of a simulated numerical example designed to provide additional insight into the behavior of our approximation procedure. Section 5 discusses the addition of a penalty associated with losing a competition and the behavior of our procedure in a particular deterministic bidding environment. For general accounts of stochastic approximation, refer to [17], [22], and [23]. We shall make use of results in [2] and [3]. Generalizations of the Kiefer-Wolfowitz procedure are discussed in [5] and [6].

The quantitative competitive bidding literature contains papers developing the entire spectrum of mathematical models from purely statistical to purely game theoretic. We have included a variety of these papers in our references; however, our approach is purely statistical and is properly considered as a descendant of the original Friedman model [8]. Unlike previously suggested statistical procedures, however, it is nonparametric, i.e., it is not necessary to specify a functional form for p. The calculations involved are trivial; they can be performed on any modern desk calculator, or for that matter, on a slide rule.

2. PRELIMINARIES

For $x \geqslant -100$, let $p(x)$ be the probability of winning any competition when the percentage markup over cost is x, i.e., given cost $C > 0$, the bid is $(1+0.01x)C$, and p is assumed to be independent of C. (By cost here, we mean the direct projected expenditures involved in fulfilling the terms of the contract.) This setup is equivalent to that of the Friedman model.

We assume that p is nonincreasing on $(-100, \infty)$ and that $b = \sup\{x : p(x) > 0\} \epsilon (0, \infty)$. It is clear that there exists, in any realistic situation, a finite least upper bound on percentage markups with any chance of winning. We do not assume that its value is known, but we do assume that $0 < b_1 \leqslant b \leqslant b_2 < \infty$ with b_1, b_2 known. In other words, we assume that the contractor is able to specify a compact interval in $(0, \infty)$ which covers b.

We shall assume that the expected percentage profit function $f(x) = xp(x)$ is strictly unimodal with unique maximum point $\theta \epsilon (0, b)$. Moreover, we assume that f is strictly increasing on $(0, \theta)$, with positive lower derivative bounded away from zero on $(0, \theta - \delta)$ for arbitrarily small $\delta > 0$, and that f is strictly decreasing on (θ, b), with negative upper derivative bounded away from zero on $(\theta + \delta, b)$ for arbitrarily small $\delta > 0$. The conditions on f are required in order to disallow such phenomena as zero-slope inflection points as well as multiple local maxima. The strict unimodality of f implies, in particular, that p must be continuous on $(0, \theta)$, left continuous at θ, and strictly decreasing on (θ, b).

3. THE APPROXIMATION PROCEDURE

Choose positive numbers X_1, a, c, and α such that $X_1 < b_1$, $c < \min(X_1, b_1 - X_1)$, $a \leqslant 2c^2/b_2$, and $3/4 < \alpha < 1$. Define positive sequences $\{a_n\}$ and $\{c_n\}$ by $a_n = an^{-\alpha}$ and $c_n = cn^{-1/4}$. With U_n^+ and U_n^- depending only on X_n, let

(1) $$P\{U_n^+ = 1 \mid X_n\} = 1 - P\{U_n^+ = 0 \mid X_n\} = p(X_n + c_n) ,$$

(2) $$V_n^+ = (X_n + c_n)U_n^+ ,$$

(3) $$P\{U_n^- = 1 \mid X_n\} = 1 - P\{U_n^- = 0 \mid X_n\} = p(X_n - c_n) ,$$

(4) $$V_n^- = (X_n - c_n)U_n^- ,$$

(5) $$Y_n = (2c_n)^{-1}(V_n^+ - V_n^-) ,$$

(6) $$X_{n+1} = X_n + a_n Y_n ,$$

(7) $$W_n = (2n)^{-1} \sum_{i=1}^{n} (U_i^+ + U_i^-) , \text{ and}$$

(8) $$Z_n = (2n)^{-1} \sum_{i=1}^{n} (V_i^+ + V_i^-) .$$

In the above setup, X_n is the estimate or approximation of θ just prior to the nth stage, and (6) is the basic recursion relation. The nth stage consists of two consecutive competitions at percentage markups X_n+c_n and X_n-c_n yielding percentage profits V_n^+ and V_n^-, respectively; U_n^+ and U_n^- are just the zero/one (lose/win) indicator random variables associated with the nth stage competitions. Y_n is the estimate of the derivative or gradient of f at X_n. (f need not be differentiable; note, however, that $E(Y_n|X_n)=(2c_n)^{-1}(f(X_n+c_n)-f(X_n-c_n))$.) The estimate moves in the estimated gradient direction with the step size and the spread between percentage markups at a stage decreasing at predetermined rates as n increases. W_n is the proportion of competitions won and Z_n is the average percentage profit over the first n stages.

Our conditions on X_1, a, and c insure that $c_n < X_n < b$ for all n. Under the assumptions of Section 2, $X_n \to \theta$ as $n \to \infty$ with probability one and in the mean square, i.e., $P\{\lim X_n=\theta\}=1$ and $\lim E[(X_n-\theta)^2]=0$ (cf. [3]). Moreover, if p is continuous at θ, then $W_n \to p(\theta)$ and $Z_n \to f(\theta)$ with probability one. Some additional assumptions yield an asymptotic distribution for X_n. If $p(\theta) < 1$, p'' exists and is continuous in a neighborhood of θ, and $f''(\theta)<0$, then $(n^{\alpha-1/2})^{1/2}(X_n-\theta)$ is asymptotically normal with mean zero and variance

$$\sigma^2=(a/4c^2)(f(\theta)(\theta-f(\theta))/|f''(\theta)|). \tag{9}$$

In other words, X_n is approximately normally distributed with mean θ and variance $\sigma^2/n^{\alpha-1/2}$ for n sufficiently large. (This result follows from a theorem in [2] since $\lim \text{var}(V_n^+|X_n)=\lim \text{var}(V_n^-|X_n)=\theta^2p(\theta)(1-p(\theta))=f(\theta)(\theta-f(\theta))$. The general theorem in [6] yields the same result under a Lipschitz condition on p'' at θ.) Of course, σ^2 is unknown, but this result does indicate the speed of convergence and the dependence of asymptotic variance upon the various parameters. It is not surprising that σ^2 is inversely related to the degree of curvature or "peakedness" of f at θ.

The asymptotic performance of the procedure obviously improves as c is increased, as a is decreased, and as α is increased. (We have disallowed $\alpha=1$ because a special condition required in this case for asymptotic normality will not generally obtain under our assumptions.) Assuming that c is specified, the magnitude of step size is governed by a, and the rate of decrease in step size is governed by α. A procedure with relatively small a and α near one might be termed "conservative," and a procedure with relatively large a and α near three-quarters might be termed "aggressive." A conservative procedure is relatively good asymptotically, but it may not move much in the short run. An aggressive procedure is relatively bad asymptotically, but it may be preferable in the short run if the initial bias $|X_1-\theta|$ is large. A lot depends on the confidence one has in the initial estimate X_1. In the case of a seasoned competitor, this estimate may incorporate a good deal of experiential intuition and/or historical data. It may be a percentage markup that has yielded satisfactory results in the past. In such instances, conservatism may be appropriate. On the other hand, an aggressive stance may be appropriate for a relatively inexperienced competitor.

We note that there is no stopping mechanism built into the approximation procedure. It is assumed that the contractor continues the procedure indefinitely. Under our assumptions, he has no reason to do otherwise.

4. SIMULATED NUMERICAL EXAMPLE

In order to give a better idea of how the approximation procedure works, we have simulated three 25-stage runs under the assumption that $p(x)=0.8-0.04x$ for $0 \leqslant x \leqslant 20=b$, $b_1=15$, and $b_2=30$. Then $\theta=10$, $p(\theta)=0.4$, and $f(\theta)=4$. In each case, we have put $X_1=9$, $c=3$, $a=0.6$, and

$\alpha=0.76$. The random numbers employed were taken from [18] as follows: column 9, page 626 in Run 1; column 1, page 627 in Run 2; column 3, page 628 in Run 3. The results are rounded to three decimal places, although more digits were actually carried in the calculations. The runs are detailed in the Appendix. The net results are: $X_{26}=9.945$, $W_{25}=0.300$, $Z_{25}=2.877$ in Run 1; $X_{26}=10.055$, $W_{25}=0.480$, $Z_{25}=4.833$ in Run 2; $X_{26}=9.266$, $W_{25}=0.440$, $Z_{25}=3.796$ in Run 3.

These runs represent three possible (although not necessarily likely) realizations for the first 25 stages of an approximation procedure with specified parameter values in the assumed bidding environment. In each case, there is a net movement, however irregular, in the direction of θ. Actually, one should expect a lot of "wandering" in this example since the objective function is relatively "unpeaked."

5. CONCLUDING REMARKS

Suppose that the contractor wishes to associate a penalty with losing a competition. Presumably, such a penalty should vary with the size of the contract. Suppose that the penalty is a fixed percentage $\beta>0$ of the contract cost, i.e., the penalty cost is $0.01\beta C$ if the contract cost is C. Then, the contractor wishes to approximate the percentage markup θ maximizing

$$f(x)=xp(x)-\beta(1-p(x))=(x+\beta)p(x)-\beta, \tag{10}$$

for $x>-\beta$. The changes required in the approximation procedure of Section 3 are $c-\beta<X_1<b_1-c$, $V_n^+=(X_n+c_n+\beta)U_n^+-\beta$, and $V_n^-=(X_n-c_n+\beta)U_n^--\beta$.

In practice, a suitable objective value for β may be difficult to obtain. However, a positive penalty may be viewed merely as a subjective device to reflect the contractor's increasing aversion to losing contracts of increasing size. Increasing β has the conservative effect of decreasing the optimal percentage markup and of increasing the corresponding probability of winning.

In Section 2, we have made certain plausible assumptions regarding the p function which guarantee convergence of the approximation procedure. However, these assumptions are not necessary for the procedure to produce satisfactory results. Consider a deterministic bidding environment where $p(x)=1$ for $x\leq b\epsilon(0,\infty)$ and $p(x)=0$ for $x>b$. One can think of this situation as arising in two somewhat artificial ways. The contractor could have a single, fixed competitor who always bids a constant (unknown) percentage markup over the contractor's cost with ties decided in favor of the contractor. Alternatively, the contractor might be the sole bidder with a fixed (unknown) upper bound on acceptable percentage markup stipulated by the contractee. Using the procedure of Section 3 with b, b_1, and b_2 playing the same roles, it is not difficult to show that $X_n\to b$, $W_n\to 1$, and $Z_n\to b$ as $n\to\infty$. As n increases, the markups get generally closer to b, but fewer and fewer of them exceed b. Our previous remarks concerning the choice of a and α also apply here.

The practical man may question the utility of a mechanistic procedure which converges rather slowly. It is true that our procedure may be somewhat myopic in a bidding environment where full disclosure of bids and identities is the rule. Even in such an environment, however, our procedure might be useful as a final "honing" device. Regarding speed of convergence, we remark that statistical estimation procedures are usually not too fast. In terms of asymptotic variance, one can generally do no better than order n^{-1}. Finally, the contractor will have to adapt to the bidding environment in some fashion. Our procedure at least has the advantage of simplicity and of known asymptotic properties under relatively weak assumptions.

APPENDIX

TABLE 1. *Simulation Run 1*

n	X_n	U_n^+	V_n^+	U_n^-	V_n^-
1	9.000	0	0	0	0
2	9.000	0	0	0	0
3	9.000	0	0	0	0
4	9.000	0	0	0	0
5	9.000	1	11.006	1	6.994
6	9.177	0	0	0	0
7	9.177	1	11.021	1	7.332
8	9.313	0	0	1	7.530
9	9.053	1	10.785	0	0
10	9.404	1	11.091	0	0
11	9.747	0	0	0	0
12	9.747	0	0	1	8.135
13	9.518	0	0	0	0
14	9.518	0	0	0	0
15	9.518	0	0	0	0
16	9.518	0	0	0	0
17	9.518	0	0	0	0
18	9.518	1	10.974	0	0
19	9.769	1	11.206	1	8.332
20	9.833	0	0	0	0
21	9.833	1	11.235	1	8.432
22	9.893	0	0	0	0
23	9.893	0	0	0	0
24	9.893	0	0	0	0
25	9.893	1	11.234	1	8.551
26	9.945				

TABLE 2. *Simulation Run 2*

n	X_n	U_n^+	V_n^+	U_n^-	V_n^-
1	9.000	1	12.000	1	6.000
2	9.600	0	0	0	0
3	9.600	1	11.880	1	7.320
4	9.860	0	0	1	7.739
5	9.479	1	11.485	0	0
6	9.984	1	11.901	0	0
7	10.461	1	12.306	1	8.617
8	10.598	1	12.382	1	8.814
9	10.722	1	12.454	0	0
10	11.128	0	0	0	0
11	11.128	0	0	0	0
12	11.128	0	0	1	9.516
13	10.860	0	0	1	9.280
14	10.609	0	0	0	0
15	10.609	1	12.133	0	0
16	10.914	0	0	1	9.414
17	10.685	0	0	0	0
18	10.685	0	0	1	9.228
19	10.474	0	0	0	0
20	10.474	1	11.892	1	9.055
21	10.535	0	0	0	0
22	10.535	1	11.920	1	9.150
23	10.592	0	0	1	9.223
24	10.406	0	0	1	9.051
25	10.227	0	0	1	8.885
26	10.055				

TABLE 3. *Simulation Run 3*

n	X_n	U_n^+	V_n^+	U_n^-	V_n^-
1	9.000	0	0	1	6.000
2	8.400	0	0	0	0
3	8.400	1	10.680	1	6.120
4	8.660	1	10.782	1	6.539
5	8.870	1	10.876	1	6.863
6	9.046	0	0	0	0
7	9.046	1	10.890	1	7.202
8	9.183	0	0	1	7.399
9	8.927	0	0	0	0
10	8.927	0	0	0	0
11	8.927	0	0	1	7.279
12	8.712	0	0	1	7.101
13	8.512	1	10.092	0	0
14	8.785	0	0	0	0
15	8.785	1	10.310	1	7.261
16	8.862	0	0	0	0
17	8.862	0	0	0	0
18	8.862	1	10.318	1	7.405
19	8.929	1	10.365	1	7.492
20	8.993	0	0	0	0
21	8.993	1	10.394	0	0
22	9.213	0	0	0	0
23	9.213	0	0	0	0
24	9.213	1	10.568	1	7.857
25	9.266	0	0	0	0
26	9.266				

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