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Shapley Value and Income Taxation

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Abstract

Income taxation is modeled as a spatial voting game where Shapley value leads to a simple linear tax function on gross income with a single marginal tax rate and a universal basic income subsidy. Shapley net income is not generally in the game core; therefore, it is inherently unstable and subject to relentless pressure from higher gross income individuals to reduce their taxes and increase their net incomes.

Keywords

Income Taxation, Cooperative Game Theory, Shapley Value

1. Introduction

Aumann and Kurz (1977) framed taxation in an organized economy as a power game among individuals with varying gross money incomes, one where a minority of high-income individuals could threaten to withhold their productive endowments to prevent an egalitarian political outcome. Shapley value for this game yielded a linear tax function with a "universal basic income" subsidy and a single marginal tax rate greater than or equal to 50%. This result mirrored tax functions derived by Mirrlees (1971) and Sheshinski (1972) without game theory. Agnew (2021) showed that this kind of linear tax function can reduce Gini inequality significantly. Reviews of optimal taxation have been provided by Mankiw, Weinzierl, and Yotan (2009) and Diamond and Saez (2011). Calvo (2021) framed redistribution of tax resources across regions as a carefully crafted cooperative game and illustrated with data from Spain.

Here we parallel the Aumann-Kurz game-theoretic result with a different blend of economic and political power, specifically a spatial voting game with a simple quota of individuals for positive-income coalition formation. The resulting Shapley value tax function has the same linear form with a common subsidy and a single marginal tax rate. There is a caveat to the stability of this solution, however, in terms of domination and the core.

In the next section, we review requisite elements of cooperative game theory from reference Owen (2001). The following section contains our main results.

2. Cooperative Games and Shapley Value

We have a finite set of players $N=\{1,\cdots,n\}$ and a superadditive characteristic function v defined on subsets of N with $v(\phi)=0$. Nonempty subsets of N are called coalitions. Superadditivity requires that $v(S\cup T)\geq v(S)+v(T)$ whenever $S\cap T=\phi$. We assume throughout that v reflects transferable currency. A game is essential if $v(N)>\sum_{i\in N}v(\{i\})$ and inessential otherwise. An inessential game is not really a negotiation game at all because every player is on his own. Nevertheless, it will be useful for us to start with just such a game in the next section before layering on additional structure. We assume throughout that $n\geq 2$.

An imputation is a vector $x=(x_1,\cdots,x_n)$ such that $x_i\geq v(\{i\})$ and $\sum_{i=1}^{n}x_i=v(N)$. The idea here is that a player must receive at least what he can achieve on his own and that the grand coalition N will ultimately form so the issue is fair distribution of the total pie. There are various solution concepts for cooperative games in terms of imputations but we will focus on two, Shapley value and the core.

We say that imputation y dominates imputation z if $y_i>z_i$ for all i in some nonempty $S\subset N$ and $\sum_{i\in S}y_i\leq v(S)$. The core $C(v)$ is the set of undominated imputations and is characterized as the set of imputations x satisfying $\sum_{i\in S}x_i\geq v(S)$ for all $S\subset N$ and $\sum_{i\in N}x_i=v(N)$. If the core is nonempty, imputations outside of it are inherently unstable.

Shapley value is the particular imputation $\phi[v]$ defined for characteristic function v by $\phi_i[v]=\sum_{\substack{S\subset N\\ i\in S}}\frac{(s-1)!(n-s)!}{n!}\left[v(S)-v(S-\{i\})\right]$ where $s=|S|$ = number of elements in set S and $\gamma(S)=\frac{(s-1)!(n-s)!}{n!}=\frac{1}{n\binom{n-1}{s-1}}$ depends only on the size of S. Shapley value is derived axiomatically but it has a heuristic expected value interpretation involving randomly permuted arrivals, all with the same probability $1/n!$. If player i arrives and finds coalition $S-\{i\}$ already there, he receives his marginal value $v(S)-v(S-\{i\})$. Shapley value $\phi_i[v]$ is the expected payoff to player i under this randomization scheme. Shapley value is widely viewed as distributionally fair, but it is not generally in the core.

3. Simple Economic Game

Consider a population of n individuals, a nonnegative vector $y=(y_1,\cdots,y_n)$ of gross (pre-tax) incomes, and characteristic function defined on subsets of *N* by $v(S)=\sum_{i\in S} y_i$ for all nonempty $S\subset N$. This game is inessential in that every individual is on his own. Think of a collection of self-sufficient Robinson Crusoe style farmers. Not surprisingly, Shapley value for this game is defined by $\phi_i[v]=\sum_{\substack{S\subset N\\ i\in S}}\gamma(S)y_i = y_i\sum_{s=1}^{n}\frac{\binom{n-1}{s-1}}{n\binom{n-1}{s-1}}=y_i$ because there are $\binom{n-1}{s-1}$ ways to choose the elements of set *S* of size *s* containing *i*. In the absence of any cooperation, each individual *i* simply gets his own output = gross income = y_i.

Now suppose that the individuals in this simple economy decide to form a "society" to provide mutual defense, dispute resolution, and other basic shared community services. For simplicity, we ignore the cost of government as it is easy to add. Our focus now turns to fair income sharing with additional cooperative structure.

As an inducement to cooperation, assume that the individual citizen-producers, through their government, agree that no one will keep any income unless he enters a coalition with at least *m* members. For $m\ge 2$, this defines an essential game *u* with $u(S)=\sum_{i\in S} y_i$ if $|S|\ge m$ and $u(S)=0$ otherwise, where $m\in N$ represents a cooperation quota or threshold. This is a spatial voting game as defined in Owen (2001), Chapter 16. If *m* = 1, we again have the original laissez-faire game with a minimal government veneer. If *m* = *n*, we have a commune. Most interesting situations are between these two extremes and in principle any level of cooperation is possible. We proceed to derive Shapley value for this game with $z_i=\phi_i[u]$. We assume throughout that $0\le y_1\le y_2\le\cdots\le y_n$ and that $y_1<y_n$.

Proposition 1. Shapley net (post-tax) income for game *u* is defined by $z_i=(1-r)y_i+r\overline{y}$ with linear tax $t_i=y_i-z_i=ry_i-r\overline{y}$ where $\overline{y}=\sum_{i=1}^{n} y_i/n$ is the overall mean gross income, $r=\frac{m-1}{n-1}$ is a single marginal tax rate, and $r\overline{y}$ is a universal basic income subsidy.

Proof. Assume $m\ge 2$ throughout since the assertion is trivially true for *m* = 1. We have

$$z_i=\sum_{\substack{S\subset N\\ i\in S}}\gamma(S)\left[u(S)-u(S-\{i\})\right]=\sum_{\substack{S\subset N\\ i\in S\\ |S|>m}}\frac{y_i}{n\binom{n-1}{s-1}}+\sum_{\substack{S\subset N\\ i\in S\\ |S|=m}}\frac{1}{n\binom{n-1}{m-1}}\sum_{j\in S} y_j$$

$$=\sum_{\substack{S\subset N\\ i\in S\\ |S|\ge m}}\frac{y_i}{n\binom{n-1}{s-1}}+\sum_{\substack{S\subset N\\ i\in S\\ |S|=m}}\frac{1}{n\binom{n-1}{m-1}}\sum_{\substack{j\in S\\ j\ne i}} y_j = y_i\sum_{s\ge m}\frac{\binom{n-1}{s-1}}{n\binom{n-1}{s-1}}+\frac{\binom{n-2}{m-2}}{n\binom{n-1}{m-1}}\sum_{j\ne i} y_j$$

where $\binom{n-1}{s-1}$ = number of ways to form a set S of size s including i and $\binom{n-2}{m-2}$ = number of ways to form a set S of size m including i and also a particular $j \neq i$. Hence, $z_i = \left(\frac{n-(m-1)}{n}\right) y_i + \frac{m-1}{n(n-1)} \sum_{j \neq i} y_j = \left(1 - \frac{m-1}{n}\right) y_i$

$+ \frac{m-1}{n(n-1)} \left(\sum_{j=1}^{n} y_j - y_i \right) = \left(1 - \frac{m-1}{n-1}\right) y_i + \left(\frac{m-1}{n-1}\right) \bar{y}$ and the assertion follows. ▪

Under this Shapley structure, the tax bill for individual i is $t_i = y_i - z_i = r(y_i - \bar{y})$ with common marginal tax rate $r = \frac{m-1}{n-1}$ applied to $y_i - \bar{y}$. This results in a positive tax bill if $y_i > \bar{y}$ and a negative tax bill if $y_i < \bar{y}$. Total income remains the same since $\sum_{i=1}^{n} z_i = u(N) = \sum_{i=1}^{n} y_i$ and the rank ordering of gross and net income remains the same. Moreover, individual tax rate $t_i / y_i = r(1 - \bar{y}/y_i)$ increases progressively with y_i and approaches r for very large gross incomes.

Proposition 2. For $m = 1$, $z = y$ is the unique element of $C(u) = C(v)$. For $m = n$, y and z are both elements of $C(u)$, along with every other imputation. For $2 \le m \le n-1$, $y \in C(u)$ but $z \notin C(u)$.

Proof. For all $m \in N$, $y \in C(u)$ because $\sum_{i \in S} y_i \ge u(S)$ for all $S \subset N$ and $\sum_{i \in N} y_i = u(N)$. For $m = 1$, y is the unique core element because any other imputation $x \in C(u) = C(v)$ must have $x_i \ge y_i$ for all i and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$; there is only one such imputation. For $m = n$, $z_i = \bar{y}$ for all i which satisfies requirements for $C(u)$ membership since $\sum_{i \in S} z_i \ge u(S) = 0$ for all proper subsets of N and $\sum_{i \in N} z_i = u(N) = \sum_{i \in N} y_i$. However, all other imputations satisfy this requirement so there is nothing unique about Shapley taxation when only the grand coalition has value. For $2 \le m \le n-1$, we can subtract a small positive amount $\varepsilon \le z_1$ from the lowest income individual and apportion it to all higher income individuals to form a new imputation $x_1 = z_1 - \varepsilon$, $x_i = z_i + \frac{\varepsilon}{n-1}$ for $i > 1$ so that $\sum_{i \in N} x_i = \sum_{i \in N} z_i = \sum_{i \in N} y_i$. Then imputation x dominates imputation z through $S = \{2, \cdots, n\}$ if $\sum_{i=2}^{n} x_i \le u(S) = \sum_{i=2}^{n} y_i$ which is true if

$$\sum_{i=2}^{n} \left\{ \left(1 - \frac{m-1}{n-1}\right) y_i + \left(\frac{m-1}{n-1}\right) \bar{y} + \frac{\varepsilon}{n-1} \right\} \le \sum_{i=2}^{n} y_i$$ and this reduces to

$$\varepsilon \le \left(\frac{m-1}{n-1}\right)(n\bar{y} - y_1) - (m-1)\bar{y} = \left(\frac{m-1}{n-1}\right)(\bar{y} - y_1) > 0$$. Hence, for sufficiently small ε imputation x dominates the Shapley imputation z through coalition

$S = \{2, \cdots, n\}$ for stipulated m and therefore $z \notin C(u)$. ▪

In addition to this proof, we observe that the gross income imputation y dominates the Shapley net income imputation z through set $S = \{i : y_i > \overline{y}\}$ if $m \le |S|$. Hence, a lower cooperation threshold makes it easier for the Shapley net income imputation to be dominated by higher income individuals without enlisting support from anyone with below average gross income.

4. Conclusion

Shapley value is utilized for fair apportionment of value in a wide variety of economic and political settings. In the income taxation setting, with a simple spatial voting game construct, it yields an appealing linear tax function with a single marginal tax rate and a universal basic income subsidy. In general, however, Shapley net income is unstable because it is not in the core. Consequently there will be unrelenting pressure from higher income individuals to reduce their taxes and increase their net incomes. This is what theory indicates and this is what we observe in reality. Despite inequality reduction and societal benefits, it is inherently difficult to establish and maintain income sharing through taxation.

Conflicts of Interest

No potential conflict of interest was reported by the author.

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