Then from (7) we have

$$E_n - E_{n-1} = \frac{(-1)^{n-1}}{2n-1}, \qquad O_n - O_{n-1} = \frac{(-1)^{n-1}}{2n-2}.$$

From (4) we get

$$E_1 = I_2 = 1 - \frac{\pi}{4}, \qquad O_1 = I_1 = \frac{1}{2}\log 2.$$

Consequently,

$$E_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k-1} - \frac{\pi}{4} \tag{9}$$

and

$$O_n = \frac{1}{2}\left(\log 2 - \sum_{k=1}^{n-1} \frac{(-1)^{k-1}}{k}\right). \tag{10}$$

Thus the results (1) and (2) follow from (6), (8), (9), and (10).


REFERENCE

1. Konrad Knopp, *Theory and Application of Infinite Series*, Blackie & Son Limited, London and Glasgow, 1957.



*Department of Mathematics and Physics, Anshan Institute of Iron and Steel Technology, Anshan 114002, Liaoning, People's Republic of China*
*Lewzheng@163.net*


# On the TEAM Approach to Investing


**Robert A. Agnew**


**1. INTRODUCTION.** In a recent article in this MONTHLY, Gerth introduced the Target Equity Allocation Management (TEAM) portfolio strategy and proved that any Buy and Hold (BH) strategy is dominated by a TEAM strategy with different initial allocations between a cash fund and a stock fund [**1**]. This dominance, however, is at a specified future time period, and the TEAM strategy, which can be highly leveraged on the front end, requires unlimited borrowing capacity along the way. This issue has been discussed in the finance literature [**3**].

We provide an alternative proof of Gerth's result that makes the dominating TEAM portfolio explicit. Then, we investigate the risk of insolvency under TEAM in the context of a plausible example. We conclude that insolvency risk is an issue only for investment horizons above 15 years. Transaction costs and taxes are ignored throughout.

For any random variable $X$, we let $E(X)$ and $\mathrm{Var}(X)$ denote the expected value and variance of $X$, respectively. Following Gerth, we define the auxiliary function

$$f(X) = \frac{E(X)}{\sqrt{\mathrm{Var}(X)}},$$

which is the reciprocal of the coefficient of variation for non-constant $X$.

**2. TEAM DOMINANCE.** We have initial portfolio value $V_0$ with initial allocations $C_0$ and $S_0 > 0$ to the cash fund and stock fund, respectively ($C_0 + S_0 = V_0$). In the BH case, over an $n$ period horizon, the future value of the cash fund is $C_n = C_0(1+c)^n$ for some fixed $c > 0$, and the future value of the stock fund is $S_n = S_0 \prod_{i=1}^{n}(1+s_i)$, where the $s_i$ are independent and identically distributed random variables with mean $\mu > c$ and variance $\sigma^2 > 0$. Future portfolio value is then $V_n = C_n + S_n$.

The TEAM strategy incorporates a "sweep" feature that makes $S_i = S_{i-1}(1+c)$ at each stage. Hence if $s_i > c$, funds are swept out of stock and into cash. If $s_i < c$, funds are swept out of cash and into stock. Gerth shows that for the same initial allocations to cash and stock, the TEAM strategy gives future portfolio value, at the end of the $n$ period horizon, with smaller mean and variance than the BH strategy, but that there exists a different initial allocation $C_0'$ and $S_0'$ such that the TEAM strategy gives future portfolio value with larger mean and the same variance. Nevertheless, it is important to recognize that we are talking about a comparison in a *fixed* future time period and furthermore that the initial TEAM allocation may involve significant leverage, i.e., $S_0' > V_0$ and $C_0' < 0$, with the latter representing borrowed cash.

Following Gerth, we distinguish the future portfolio values of the BH and TEAM strategies after $n$ periods as $B_n$ and $T_n$, respectively, given the same initial allocations $C_0$ and $S_0$. Putting $R_i = (s_i - c)/(1+c)$ and $M_n = V_0(1+c)^n$, Gerth shows that if

$$T_n - M_n = S_0(1+c)^n \sum_{i=1}^{n} R_i,$$

then

$$E(T_n - M_n) = S_0(1+c)^{n-1} n(\mu - c),$$
$$\mathrm{Var}(T_n - M_n) = \mathrm{Var}(T_n) = S_0^2(1+c)^{2(n-1)} n\sigma^2,$$

and hence that

$$f(T_n - M_n) = \frac{E(T_n - M_n)}{\sqrt{\mathrm{Var}(T_n - M_n)}} = \sqrt{n}\left(\frac{\mu - c}{\sigma}\right),$$

independent of $S_0$. He also shows that $f(B_n - M_n) < f(T_n - M_n)$ for $n \geq 2$ and that there exists an alternative initial allocation $C_0'$ and $S_0'$ and terminal value $T_n'$ with $E(T_n' - M_n) > E(B_n - M_n)$ while $\mathrm{Var}(T_n' - M_n) = \mathrm{Var}(B_n - M_n)$, but he doesn't compute the required adjustment $S_0'/S_0$. We do so now. The following BH formulas are well known in the finance literature [**2**], [**3**], [**4**, p. 110].

**Lemma 1.** *$E(B_n - M_n) = S_0((1+\mu)^n - (1+c)^n)$, $\mathrm{Var}(B_n - M_n) = \mathrm{Var}(B_n) = S_0^2((\sigma^2 + (1+\mu)^2)^n - (1+\mu)^{2n})$, and hence*

$$f(B_n - M_n) = \frac{E(B_n - M_n)}{\sqrt{\mathrm{Var}(B_n - M_n)}} = \frac{1 - \left(\frac{1+c}{1+\mu}\right)^n}{\sqrt{\left(\left(\frac{\sigma}{1+\mu}\right)^2 + 1\right)^n - 1}},$$

*independent of* $S_0$.

*Proof.* $E(\prod_{i=1}^n (1 + s_i)) = \prod_{i=1}^n E(1 + s_i) = (1 + \mu)^n$ and

$$\begin{aligned}\mathrm{Var}\left(\prod_{i=1}^n (1 + s_i)\right) &= E\left(\left(\prod_{i=1}^n (1 + s_i)\right)^2\right) - \left(E\left(\prod_{i=1}^n (1 + s_i)\right)\right)^2 \\ &= \prod_{i=1}^n \left(\mathrm{Var}(1 + s_i) + (E(1 + s_i))^2\right) - (1 + \mu)^{2n} \\ &= (\sigma^2 + (1 + \mu)^2)^n - (1 + \mu)^{2n}\end{aligned}$$

The assertion now follows since $B_n - M_n = S_0(\prod_{i=1}^n (1 + s_i) - (1 + c)^n)$. ■

**Lemma 2.** *For* $n \geq 2$, $f(B_n - M_n) < f(T_n - M_n)$.

*Proof.* Strict convexity implies $g(t) = t^n - 1 > n(t - 1)$ for any $t > 0, t \neq 1$, so that

$$1 - \left(\frac{1 + c}{1 + \mu}\right)^n < n\left(1 - \frac{1 + c}{1 + \mu}\right) = n\left(\frac{\mu - c}{1 + \mu}\right),$$

$$\left(\left(\frac{\sigma}{1 + \mu}\right)^2 + 1\right)^n - 1 > n\left(\frac{\sigma}{1 + \mu}\right)^2,$$

and hence that

$$\frac{1 - \left(\frac{1+c}{1+\mu}\right)^n}{\sqrt{\left(\left(\frac{\sigma}{1+\mu}\right)^2 + 1\right)^n - 1}} < \sqrt{n}\left(\frac{\mu - c}{\sigma}\right).$$ ■

**Proposition 3 (Gerth).** *If* $n \geq 2$ *and*

$$\frac{S_0'}{S_0} = \frac{\sqrt{(\sigma^2 + (1 + \mu)^2)^n - (1 + \mu)^{2n}}}{(1 + c)^{n-1}\sqrt{n}\sigma},$$

*then the resulting TEAM strategy with initial stock allocation* $S_0'$ *and terminal value* $T_n'$ *has* $E(T_n') > E(B_n)$ *and* $\mathrm{Var}(T_n') = \mathrm{Var}(B_n)$.

*Proof.* By construction $\mathrm{Var}(T_n') = \mathrm{Var}(B_n)$. Moreover, $f(T_n' - M_n) > f(B_n - M_n)$ independent of the initial allocations. Hence,

$$E(T_n' - M_n) = f(T_n' - M_n)\sqrt{\mathrm{Var}(T_n')} > f(B_n - M_n)\sqrt{\mathrm{Var}(B_n)} = E(B_n - M_n).$$ ■

**3. RISK OF TEAM INSOLVENCY.** By insolvency, we mean that investment value becomes negative. Bad things happen when an investment portfolio becomes insolvent, e.g., a margin call or a loan recall, even for hedge funds or other large players, as we have seen in recent years.

There is no insolvency risk associated with BH, assuming that $0 < S_0 \leq V_0$. With TEAM, there is certainly insolvency risk if $S_0' > V_0$, and possibly when $S_0' \leq V_0$, because of the sweep feature. The probability of TEAM insolvency is $P \cup_{k=1}^{n} \{T_k' < 0\}$, i.e., the probability that some $T_k'$ becomes negative across the investment horizon. Recall that $T_k' = V_0(1+c)^k + S_0'(1+c)^k Z_k$ for all $k = 1, \ldots, n$, where $Z_k = \sum_{i=1}^{k} R_i$. Hence, $P \cup_{k=1}^{n} \{T_k' < 0\} = P \cup_{k=1}^{n} \{Z_k < -\alpha\}$, where $\alpha = V_0/S_0'$. Since the $Z_k$ are sums of independently and identically distributed random variables, we assume that the $Z_k$ are approximately normally distributed for simulation purposes, which is supported by the Central Limit Theorem when $k$ is large. We have

$$E(Z_k) = k\left(\frac{\mu - c}{1+c}\right) \quad \text{and} \quad \sqrt{\mathrm{Var}(Z_k)} = \sqrt{k}\left(\frac{\sigma}{1+c}\right).$$

$P \cup_{k=1}^{n} \{Z_k < -\alpha\}$ is not easy to compute exactly, but it is easy to estimate via Monte Carlo. Moreover, we have natural computable bounds

$$\max_{1 \leq k \leq n} P\{Z_k < -\alpha\} \leq P \bigcup_{k=1}^{n} \{Z_k < -\alpha\} \leq \sum_{k=1}^{n} P\{Z_k < -\alpha\}.$$

From data in [**4**], we have the historical statistics displayed in Table 1.

TABLE 1. Historical percentage return statistics.

| | Stocks | | T-Bills | |
|---|---|---|---|---|
| Period | Mean | Std Deviation | Mean | Std Deviation |
| 1949–1998 | 14.8 | 16.6 | 5.1 | 3.0 |
| 1969–1998 | 13.9 | 16.5 | 6.8 | 2.6 |
| 1979–1998 | 18.5 | 13.1 | 7.2 | 3.0 |

Looking forward, we adopt illustrative annual estimates of $\mu = 15\% = .15$, $\sigma = 15\% = .15$, and $c = 5\% = .05$, ignoring volatility in the cash rate. We further assume a 50-50 BH portfolio with $V_0 = 1$ and $C_0 = S_0 = 0.5$. Our Monte Carlo estimates reflect 30,000 multi-year trials yielding statistical precision well within one percentage point for all cases. The results are summarized in Table 2.

It is clear that insolvency risk is miniscule for investment horizon up through 10 years and is quite small for 15 years. It becomes more significant at 20 years and especially at 25 years, where the TEAM portfolio is highly leveraged on the front end. Still, the risk is only about 7%, while the TEAM terminal expected value is more than double that of BH's. One can imagine a high-stakes player taking on such a risk, but lenders would undoubtedly demand a higher rate to reflect the risk, which in turn would diminish TEAM advantage and increase insolvency risk further. This effect could be captured only by a more complex model.

We conclude that the TEAM portfolio strategy provides significant benefit over a moderate investment horizon with insignificant insolvency risk. For a longer horizon, leverage and insolvency risk increase and the model assumptions become more ten-

TABLE 2. Estimates of TEAM insolvency risk over multiyear horizon.

$\mu = \sigma = .15, \quad c = .05, \quad C_0 = S_0 = 0.5$

| Investment Horizon ($n$) | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|
| $S_0'$ | .7318 | 1.178 | 1.898 | 3.058 | 4.929 |
| $E(T_n')$ | 1.721 | 3.457 | 7.716 | 18.11 | 43.13 |
| $E(B_n)$ | 1.644 | 2.837 | 5.108 | 9.510 | 18.15 |
| $\sqrt{\text{Var}(T_n')} = \sqrt{\text{Var}(B_n)}$ | .2984 | .8671 | 2.183 | 5.184 | 11.92 |
| $\alpha$ | 1.366 | .8486 | .5269 | .3270 | .2029 |
| Insolvency Risk: | | | | | |
| Lower Bound | $4.01 \times 10^{-9}$ | $3.45 \times 10^{-5}$ | .00085 | .0066 | .0258 |
| Monte Carlo | 0 | $6.67 \times 10^{-5}$ | .00333 | .0220 | .0696 |
| Upper Bound | $4.50 \times 10^{-9}$ | $1.85 \times 10^{-4}$ | .00738 | .0523 | .1692 |

uous, although we can imagine a high-stakes player attempting to implement such a strategy.

REFERENCES


1. F. Gerth III, The TEAM approach to investing, *Amer. Math. Monthly* **106** (1999) 553–558.
2. W. Y. Lee, Diversification and time: Do investment horizons matter?, *J. Portfolio Mgt.* **16** (Spring 1990) 21–26.
3. H. Levy and D. Gunthorpe, Optimal investment proportions in senior securities and equities under alternate holding periods, *J. Portfolio Mgt.* **19** (Summer 1993) 30–36.
4. *Stocks, Bonds, Bills, and Inflation 1999 Yearbook*, Ibbotson Associates, Chicago, 1999.



*Discover Financial Services, Riverwoods, IL 60015-3851*
*robertagnew@discoverfinancial.com*


# The Friendship Theorem


**Craig Huneke**


The 'friendship theorem' can be stated as follows [**1**, p. 183]:

> *Suppose in a group of at least three people we have the situation that any pair of persons have precisely one common friend. Then there is always a person who is everybody's friend.*

The first published proof of this theorem of which I am aware was due to Paul Erdős, Alfred Rényi, and Vera Sós [**3**]. Translating the theorem into graph theory yields the following theorem:

**Theorem.** *If G is a graph in which any two distinct vertices have exactly one common neighbor, then G has a vertex joined to all others.*

As a consequence, such graphs are completely determined; they consist of edge-disjoint triangles around a common vertex. The best known and simplest proof is based on computing the eigenvalues (and their multiplicities) of the square of the ad-