THEOREM 2. *When $n \geq 2$ is even, integers a, b, and c satisfy $a^2 + b^2 = c^n$ if and only if*

$$a = r^{n/2}\,\mathrm{Re}(z\omega), \qquad b = r^{n/2}\,\mathrm{Im}(z\omega), \qquad c = \pm r \prod_{t=0}^{(n-2)/2} \overline{z}_t z_t \tag{4}$$

where r is a positive integer, ω is a unit, each z_t is a Gaussian integer, and $z = \prod_{t=0}^{(n-2)/2} \overline{z}_t^{\,t} z_t^{n-t}$.

Proof. The proof follows along the same lines as the previous one. The only subtlety to point out is that although 2 and n divide β_i, $2n$ might not divide β_i. However, the term $\prod_{i=1}^m q_i^{\beta_i/2} = (\prod_{i=1}^m q_i^{\beta_i/n})^{n/2}$ and the (integer) term of the form $\overline{z}_t^{n/2} z_t^{n/2}$ can be absorbed into the integer $r^{n/2}$. ■

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On the Two-Box Paradox

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On a game show, you are presented with two identical boxes. Both boxes contain positive monetary prizes, one twice the other. You are allowed to pick one box and observe the prize $x > 0$, after which you can choose to trade boxes. In terms of simple expected value, it is *always* better to trade since $\frac{1}{2}(2x) + \frac{1}{2}\left(\frac{x}{2}\right) = \frac{5x}{4} > x$. That is the paradox.

Simple thought experiments suggest that a sufficiently large observed prize would cause a player not to trade, despite the mathematical computation of expected value. In individual cases, this creates some threshold, which depends on the observed prize, for ceasing to trade. A player may have in mind prior probabilities about what prizes the game show would offer, so that an observed prize of \$10,000, for instance, would not yield equal *judgmental* odds of \$20,000 or \$5,000 in the unobserved box. The judgmental probability approach to the two-box problem seeks to develop optimal threshold strategies in terms of prior distributions on the set of possible prizes. Recent articles in this MAGAZINE have focused on the judgmental probability approach, although they have also discussed the second line of attack on this problem, expected utility [**2, 3**].

In expected utility theory, it is assumed than an individual has an underlying utility function for wealth. This utility function is increasing because it is presumed that an individual will always prefer more wealth to less wealth. In addition, the utility function is concave because it is presumed that an individual will have nonincreasing marginal utility for wealth. The utility function u is thus an increasing, concave function from the positive half line into the real line. The scaling on this function is unimportant because a positive linear transformation $a + bu$, with $b > 0$, is equivalent for individual decision making. Finally, linear utility $u(w) = w$, which is inherent in the simple statement of the two-box paradox, really represents a boundary case. Economists normally assume strictly diminishing marginal utility for wealth, for instance, a person prefers \$100,000 of wealth to \$50,000 but he has less use for an *additional* dollar when he has \$100,000 than when he has \$50,000.

To recapitulate, we assume that an individual has an underlying utility function, or preference function, for wealth, even though he or she may not have detailed this function. The expected utility hypothesis takes this notion of utility of wealth one step further. It is assumed that a rational individual will explicitly model his utility function for wealth u and will select among risky prospects based on maximal expected utility. In the context of the two-box game, it is assumed that a rational individual will trade if and only if $(1/2)u(w_0 + 2x) + (1/2)u(w_0 + x/2) > u(w_0 + x)$, where w_0 is his initial wealth and x is the observed prize. Note that we are not deviating from the 50-50 prize distribution between $2x$ and $x/2$, which is inherent in the statement of the two-box game. There is no injection of judgmental probabilities. Our subsequent analysis is based on expected utility theory alone.

Expected utility has a rich history dating back to Daniel Bernoulli's 1738 resolution of the St. Petersburg paradox. In the St. Petersburg game, a fair coin is flipped until a head occurs, and a player receives 2^n when the first head occurs at trial n. It is easy to see that expected payoff is infinite. Nevertheless, Bernoulli observed that no rational individual would pay a huge amount to play this game and he resolved the paradox by assuming logarithmic utility for wealth. Later, Karl Menger in 1934 observed that full resolution of the St. Petersburg paradox requires a utility function that is *bounded above*. A nice historical perspective is provided by Fonseca and Ussher [**4**].

Once again, this note focuses on a pure expected utility approach, without reference to any prior distribution; the only probabilistic element is the 50-50 prize distribution between $2x$ and $x/2$, which is associated with the trading gamble when a player in the two-box game is faced with two identical boxes and an observed prize of x. In that context, we show that the two-box paradox is confined to unbounded utility functions and that common bounded utility functions have well-defined optimal threshold strategies.

Utility functions and risk Assume that an individual has utility $u(w)$ for wealth $w > 0$. Economists generally assume that rationality requires the utility function u to be strictly increasing and concave ($u' > 0$ and $u'' \leq 0$ assuming differentiability). Normally, the utility function should be strictly concave to reflect a person's diminishing preference for an *additional* dollar at increasing wealth levels. Economists also generally postulate that a rational individual facing alternative risky prospects, or gambles, will choose to maximize expected utility.

Beyond these general notions of rationality, economists have defined local measures of absolute risk aversion as

$$-u''(w)/u'(w)$$

and relative risk aversion as

$$-wu''(w)/u'(w).$$

These measures follow naturally from Taylor approximations to the so-called *risk premia* associated with small random perturbations to wealth, either additive or proportional [**4, 5, 6**]. Given utility function u and wealth level w, the risk premium π associated with an additive random wealth perturbation ϵ is defined by the functional equation $u(w - \pi) = \mathrm{E}(u(w + \epsilon))$, where π is the premium that one is willing to pay to avoid the random wealth perturbation. We have

$$u(w-\pi) \cong u(w) - u'(w)\pi \quad \text{and} \quad u(w+\epsilon) \cong u(w) + u'(w)\epsilon + u''(w)\epsilon^2/2.$$

If $\mathrm{E}(\epsilon) = 0$ and $\mathrm{Var}(\epsilon) = \sigma^2$, then

$$\mathrm{E}\left(u(w+\epsilon)\right) \cong u(w) + u''(w)\sigma^2/2 \quad \text{and hence} \quad \pi \cong \left(-u''(w)/u'(w)\right)\sigma^2/2,$$

with units $\$ = (1/\$) \times \$^2$. Relative risk aversion has a similar interpretation with respect to the proportional risk premium defined by the functional equation

$$u\big((1-\pi)\cdot w\big) = \mathrm{E}\left(u\big((1+\epsilon)\cdot w\big)\right),$$

that is, π is the proportion of wealth one will pay to avoid a proportional wealth perturbation ϵ. In this case, we have dimensionless $\pi \cong (-wu''(w)/u'(w))\sigma^2/2$.

Up to simple scaling, identically zero risk aversion, either absolute or relative, implies a linear utility function $u(w) = w$. Constant *absolute* risk aversion $\alpha > 0$ implies a utility function of the form $u(w) = -e^{-\alpha w}$, which is bounded above. Once again, scaling is unimportant and the utility function $u(w) = 100(1 - e^{-\alpha w})$, which takes positive values, is completely equivalent for our purposes. The higher the risk aversion parameter α, the less tolerance the individual has for additive wealth perturbation at any level of wealth.

Up to simple scaling, constant positive *relative* risk aversion $\beta > 0$ implies a utility function of the form $u(w) = w^{1-\beta}$ for $\beta \in (0, 1)$, $u(w) = \ln(w)$ for $\beta = 1$, or $u(w) = -w^{1-\beta}$ for $\beta > 1$, where the latter function is bounded above. Once again, scaling is unimportant but we can't avoid negative utility values for small levels of wealth when $\beta \geq 1$. The higher the risk aversion parameter β, the less tolerance the individual has for proportional wealth perturbation at any level of wealth.

Someone with a linear utility function is entirely indifferent to risk in that doubling his fortune doubles his satisfaction, and of course he is subject to the two-box paradox. The other cases aren't so obvious and we deal with them in the next two sections. We don't claim that these constant risk aversion utility functions are the only ones worth considering, although they have been much discussed. Some economists have suggested that absolute risk aversion should decrease with wealth while relative risk aversion should increase [**5**].

Two-box paradox and unbounded utility Denote initial wealth by w_0 and the observed prize by x. The two-box paradox arises when an individual prefers to trade, or gamble, without regard to his initial wealth position or the observed prize. In mathematical terms, a utility function u is subject to the paradox if

$$\frac{1}{2}u(w_0 + 2x) + \frac{1}{2}u\left(w_0 + \frac{x}{2}\right) > u(w_0 + x)$$

or equivalently

$$u(w_0 + 2x) - u(w_0 + x) > u(w_0 + x) - u(w_0 + x/2)$$

for any $w_0, x > 0$.

For an individual subject to the paradox, the gain in satisfaction from doubling the observed prize (for instance, \$10,000 to \$20,000) always exceeds the loss in satisfaction from halving the observed prize (for instance, \$10,000 to \$5,000), regardless of his initial wealth position. The following result shows that this condition is not confined to linear utility.

PROPOSITION 1. *If* $u(w) = w^\gamma$ *for* $\gamma \in (0, 1]$ *or* $u(w) = \ln(w)$, *then the paradox occurs.*

Proof. For any $w_0, x > 0$,

$$\frac{(w_0+2x)^\gamma + (w_0+x/2)^\gamma}{2} - (w_0+x)^\gamma > (w_0+x)^\gamma \left(\frac{t^\gamma + t^{-\gamma}}{2} - 1 \right)$$
$$= (w_0+x)^\gamma (t^\gamma - 1)(1 - t^{-\gamma})/2 > 0,$$

where $t = (w_0+2x)/(w_0+x) > (w_0+x)/(w_0+x/2) > 1$. Moreover,

$$\frac{\ln(w_0+2x) + \ln(w_0+x/2)}{2} - \ln(w_0+x) = \ln\left(\frac{\sqrt{w_0^2 + 5w_0x/2 + x^2}}{w_0+x} \right)$$
$$> \ln\left(\frac{\sqrt{w_0^2 + 2w_0x + x^2}}{w_0+x} \right) = 0. \quad \blacksquare$$

We have just shown that conventional unbounded utility functions are subject to the paradox. We next show that the paradox occurs only for utility functions that are unbounded above. This result has already been proved [**3**], but our proof avoids any reference to prior distributions.

PROPOSITION 2. (BRAMS AND KILGOUR) *A necessary condition for the paradox is that the utility function u be unbounded above.*

Proof. Let $a_n = u(w_0 + 2^{n+1}) - u(w_0 + 2^n)$ for $n \geq 0$, and let $s_n = \sum_{k=0}^n a_k = u(w_0 + 2^{n+1}) - u(w_0 + 1)$. If the paradox arises, then a_n is a positive, increasing sequence so that $s_n \uparrow \infty$, and thus u is unbounded above. $\blacksquare$

We conclude that utility functions that are bounded above have at least the potential for simple threshold strategies for ceasing to trade in the two-box game.

Bounded utility We now show that simple threshold strategies exist for two families of bounded utilities. For the constant absolute risk aversion family, the prize threshold is independent of initial wealth. For the family of functions with constant relative risk aversion, the prize threshold is proportional to initial wealth. These results are not surprising. Constant absolute risk aversion implies the same sensitivity to additive wealth perturbation across the spectrum of existing wealth. Constant relative risk aversion, on the other hand, implies the same sensitivity to proportional wealth perturbation across the spectrum of existing wealth. Once again, the optimal prize thresholds are independent of utility function scaling.

PROPOSITION 3. *Let $u(w) = -e^{-\alpha w}$ for some $\alpha > 0$, so that absolute risk aversion is constant and positive. Then the optimal threshold strategy in the two-box game is to trade when the observed prize x is less than $x^\star = -2\ln(\theta)/\alpha$, where $\theta = (\sqrt{5}-1)/2$, and not to trade when $x \geq x^\star$. The optimal threshold $x^\star$ is independent of initial wealth w_0.*

Proof. For $w_0, x > 0$,

$$\frac{1}{2}u(w_0+2x) + \frac{1}{2}u(w_0+x/2) - u(w_0+x) = -\frac{1}{2}e^{-\alpha(w_0+x/2)} f(t),$$

where $t = e^{-\alpha x/2}$ and $f(t) = t^3 + 1 - 2t$ (graphed in FIGURE 1).

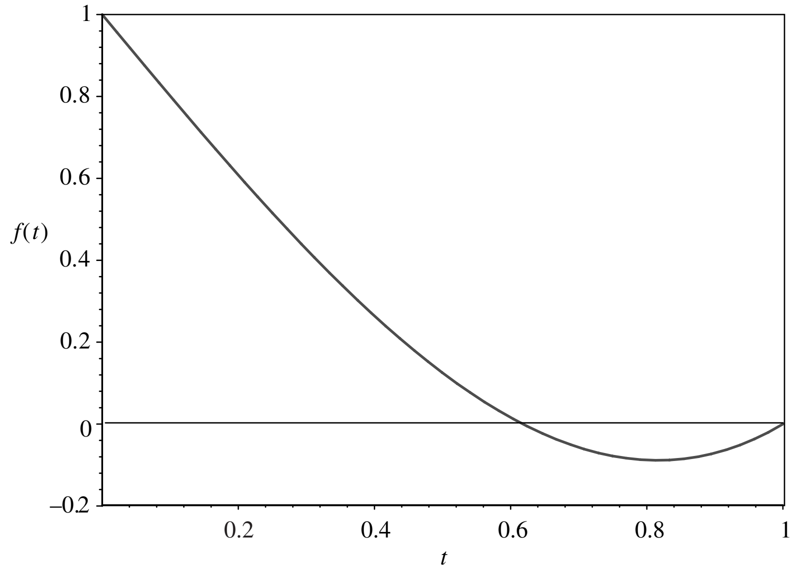

Figure 1 The function $f(t) = t^3 + 1 - 2t$

It is easy to check that f is strictly convex ($f'' > 0$) on $(0, 1)$ with $f(0) = 1$, $f(1) = 0$, $f'(0) < 0$, and $f'(1) > 0$. The relation $\theta^2 = 1 - \theta$ implies that $f(\theta) = 0$ and we conclude that θ is the unique zero of f in $(0, 1)$, with $f(t) > 0$ for $t \in (0, \theta)$ and $f(t) < 0$ for $t \in (\theta, 1)$. Since $-(1/2)e^{-\alpha(w_0+x/2)} f(t)$ is positive if and only if $t > \theta$, which is the same as $x < x^\star$, it follows that $x^\star$ is the optimal threshold point, independent of w_0. ∎

PROPOSITION 4. *Let $u(w) = -w^{-\gamma}$ for some $\gamma > 0$ so that relative risk aversion $\beta = \gamma + 1$ is constant and greater than one. Then, the optimal threshold strategy in the two-box game is to trade when the observed prize $x < x^\star = w_0\phi/(1 - \phi)$, where w_0 is initial wealth and ϕ is the unique root in $(0, 1)$ of $(1 + t)^{-\gamma} + (1 - t/2)^{-\gamma} = 2$, and not to trade when $x \geq x^\star$. The optimal threshold $x^\star$ is proportional to initial wealth w_0.*

Proof. For $w_0, x > 0$,

$$\frac{1}{2}u(w_0 + 2x) + \frac{1}{2}u(w_0 + x/2) - u(w_0 + x) = -\frac{1}{2}(w_0 + x)^{-\gamma} f(t),$$

where this time $t = x/(w_0 + x)$ and $f(t) = (1 + t)^{-\gamma} + (1 - t/2)^{-\gamma} - 2$ (graphed in FIGURE 2 for $\gamma = 1$).

It is easy to check that f is strictly convex ($f'' > 0$) on $(0, 1)$ with $f(0) = 0$, $f(1) = 2^{-\gamma} + 2^{\gamma} - 2 = (2^{\gamma} - 1)(1 - 2^{-\gamma}) > 0$, $f'(0) < 0$, and $f'(1) > 0$. We conclude that there exists a unique zero ϕ of f in $(0, 1)$ with $f(t) < 0$ for $t \in (0, \phi)$ and $f(t) > 0$ for $t \in (\phi, 1)$. Since $-(1/2)(w_0 + x)^{-\gamma} f(t)$ is positive if and only if $t < \phi$, which is the same as $x < x^\star$, it follows that $x^\star$ is the optimal threshold point and that it is proportional to w_0. ∎

We remark that an extension of Proposition 4 is easily obtained for a utility function of the form $u(w) = -(\eta + w)^{-\gamma}$ where $\gamma > 0$ and $\eta > 0$. The proof goes through the same way with optimal threshold $x^\star = (\eta + w_0)\phi/(1 - \phi)$. This utility function, which falls into the category that Gollier [**5**] calls *harmonic absolute risk aversion*, exhibits decreasing absolute risk aversion $(\gamma + 1)/(\eta + w)$ and increasing relative risk aversion $(\gamma + 1)w/(\eta + w)$, although both types of risk aversion are reduced by in-

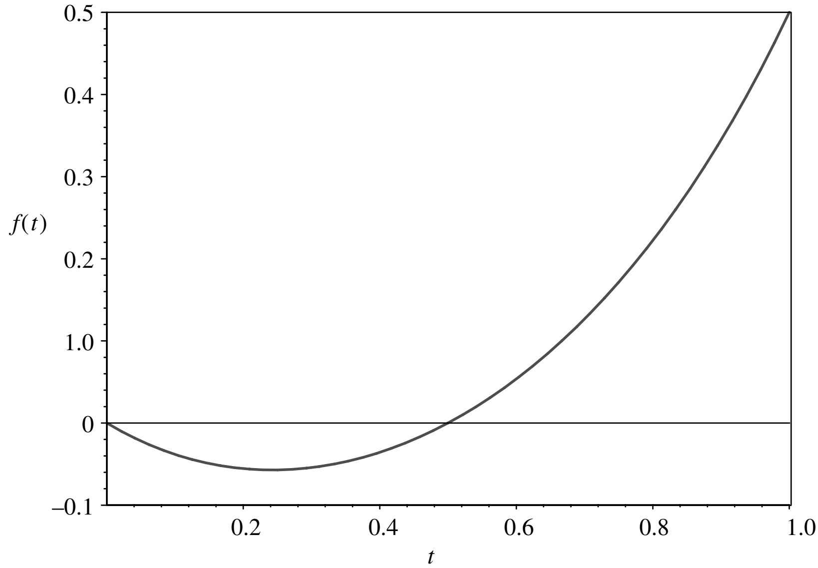

Figure 2 The function $f(t) = (1+t)^{-\gamma} + (1-t/2)^{-\gamma} - 2$

clusion of the additional parameter. The following two examples illustrate further the distinct behavioral differences that are implied by the two families of utility functions in Propositions 3 and 4, constant absolute risk aversion and constant relative risk aversion.

EXAMPLE 1. *Suppose an individual has utility function* $u(w) = 100(1 - e^{-.0001w})$, *which exhibits constant absolute risk aversion with parameter* $\alpha = .0001$. *Then, he will trade if and only if the observed prize is less than \$9,624.24, regardless of his initial wealth. If this person observes a prize of \$5,000, he will trade, no matter whether his existing wealth is \$10 or \$1,000,000. If, on the other hand, he observes a prize of \$10,000, he will not trade under any circumstance.*

EXAMPLE 2. *Suppose an individual has utility function* $u(w) = 100 - 10000w^{-1}$, *which exhibits constant relative risk aversion with parameter* $\beta = 2$. *Then, he will trade if and only if the observed prize is less than his initial wealth since the root* $\phi = 1/2$. *If this person observes a prize of \$10,000, he will trade if his existing wealth is less than \$10,000 but he will not trade if his existing wealth is \$10,000 or greater.*

In conclusion, we have demonstrated that the two-box paradox, like the St. Petersburg paradox, can be resolved by bounded utility of wealth and that for traditional bounded utility functions, simple threshold strategies are optimal.

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A Conceptual Proof of Cramer's Rule

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THEOREM. (CRAMER'S RULE) *Let A be an invertible $n \times n$ matrix. Then the solutions x_i to the system $A\mathbf{x} = \mathbf{b}$ are given by*

$$x_i = \frac{\det(A_i)}{\det(A)}, \tag{1}$$

where A_i is the matrix obtained from A by replacing the ith column of A by $\mathbf{b}$.

Proof. The classical way to solve a linear equation system is by performing row operations: (i) add one row to another row, (ii) multiply a row with a nonzero scalar and (iii) exchange two rows. We show that the quotient in equation (1) will not change under row operations.

Under the first row operation, the values of the two determinants $\det(A_i)$ and $\det(A)$ will not change, since determinants are invariant under this row operation. Under the second row operation both determinants will gain the same factor, which cancels in the quotient. Finally, under the third row operation both determinants will switch sign, which again cancels in the quotient.

Since every invertible matrix A can be row reduced to the identity matrix, it is now enough to prove Cramer's rule for the identity matrix. However, this is a straightforward task. ∎

A Parent of Binet's Formula?

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The famous Binet formula for the Fibonacci sequence $F_1 = 1 = F_2$, $F_{n+2} = F_n + F_{n+1}$ is the identity

$$F_n = \frac{\phi^n - (-1/\phi)^n}{\sqrt{5}},$$

where ϕ is the golden ratio $(1 + \sqrt{5})/2$.